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Contents

Certain Generalization of Appell's Functions and Riemann–Liouville Fractional Derivative Operator and Their Applications

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Abstract Here, the Riemann–Liouville fractional derivative operator concept and a new and interesting extended form of Appell's functions are discussed. We discovered new formulae for fractional derivatives of various well-known functions in terms of new extended Appell's hypergeometric functions of two variables and Lauricella hypergeometric functions of three variables, with a view toward the analytic properties and application of the new Riemann–Liouville-type fractional derivative operator. Additionally, we defined the Mellin transformations of that function. Next, we created generating functions for generalized extended hypergeometric functions to validate our new operator using an extended Riemann–Liouville fractional derivative operator and a new definition of extended Appell's functions.

1 Introduction and Preliminaries

Many generalizations and extensions of special functions and mathematical physics have been investigated in the past several years. In particular, special functions involving more than one variable provide a new variety of mathematical techniques for resolving huge classes of partial differential equations that are frequently encountered in physical problems. Most mathematical physics, special functions, and their generalizations are motivated by physical considerations. In which Riemann–Liouville fractional derivative operator plays an important role. The Riemann–Liouville fractional derivative operator offers a powerful mathematical tool for capturing the behavior of non-integer order systems, incorporating memory effects, and describing processes with fractal and self-similar properties. Its wide range of applications and its compatibility with existing mathematical frameworks make it a valuable tool for researchers and practitioners in various disciplines. Motivated by their importance in various fields, recently Özarslan et al. [16] studied the extended Appell's func-

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tion using the extended beta function given by Chaudhary et al. [4]. He defined the Riemann–Liouville fractional derivative operator and found the formula of some well-known functions. Furthermore, he defined their Millen transform and some generating relations for extended hypergeometric functions via a generalized fractional derivative operator. In 2022, Sadek [19–21] gave various ideas related to fractional derivative operators and related problems like fractional backward differentiation formulas for solving fractional differential matrix equations, controllability, and observability for fractal linear dynamical systems and stability of conformable linear infinite-dimensional systems and stability of conformable linear infinite-dimensional systems. After that, in 2023, Sadek et al. [22, 23] gave another idea for a new type of fractional derivatives involving exponential cotangent function in their kernels and also discussed their applications and a new numerical approach to solving the fractional differential Riccati equations (FDRC) numerically. Khan et al. [7] studied the extended beta function with the help of the Mittag-Leffler function in two parameters and discussed their application. Inspired by the work mentioned earlier, in this paper, we will study the further generalization of Appell's functions, and the Riemann–Liouville fractional derivative operator using extended beta functions given by khan et al. [7].

Throughout the paper, the set of all real numbers is represented by the symbol $\mathbb{R}$. The set of natural numbers is represented by the symbol $\mathbb{N}$. The set of complex numbers by the symbol $\mathbb{C}$. The set of positive real numbers is represented by the symbol $\mathbb{R}^+$ and the set of non-negative real numbers by the symbol $\mathbb{R}_0^+$. Due to importance of special functions many authors (see more details in [6–12, 15, 17]) have recently developed extensions to the Euler beta function, gamma function, Gauss hypergeometric function, confluent hypergeometric function, and many other functions. Below, we mentioned some well-known helpful definitions as we were required to discuss our main problem as follows:

Definition 1 The Euler beta function is defined as [2, 4]:

$$\mathcal{B}(\kappa_1, \kappa_2) = \int_0^1 \tau^{\kappa_1 - 1} (1-\tau)^{\kappa_2 - 1} d\tau,$$

$$(\Re(\kappa_1) > 0,\ \Re(\kappa_2) > 0).$$

The classical Gauss hypergeometric function $F(\kappa_1, \kappa_2; \kappa_3; z)$ and confluent hypergeometric function $\Phi(\kappa_2; \kappa_3; z)$ are defined as [18]

$$F(\kappa_1, \kappa_2; \kappa_3; z) = \frac{1}{\mathcal{B}(\kappa_2, \kappa_3 - \kappa_2)} \int_0^1 \tau^{\kappa_2 - 1} (1-\tau)^{\kappa_3 - \kappa_2 - 1} (1 - z\tau)^{-\kappa_1}\ d\tau, \quad (1)$$

$$(\ |\arg(1-z)| < \pi;\ \Re(\kappa_3) > \Re(\kappa_2) > 0\ ),$$

$$\Phi\left(\kappa_2;\kappa_3;z\right)=\frac{1}{\mathcal{B}(\kappa_2,\kappa_3-\kappa_2)}\int_0^1\tau^{\kappa_2-1}\left(1-\tau\right)^{\kappa_3-\kappa_2-1}exp\left(z\tau\right)d\tau, \tag{2}$$

$$\left(\,\Re(\kappa_3)>\Re(\kappa_2)>0\,\right).$$

By using the series expansion of $(1-z\tau)^{-\kappa_1}$ and $\exp(z\tau)$ in (1) and (2), respectively, the hypergeometric and confluent hypergeometric functions are written in terms of beta function as

$$F(\kappa_1,\kappa_2;\kappa_3;z)=\sum_{n=0}^{\infty}(\kappa_1)_n\,\frac{\mathcal{B}(\kappa_2+n,\kappa_3-\kappa_2)}{\mathcal{B}(\kappa_2,\kappa_3-\kappa_2)}\,\frac{z^n}{n!}, \tag{3}$$

$$\left(\,|z|<1,\ \Re(\kappa_3)>\Re(\kappa_2)>0\right),$$

and

$$\Phi\left(\kappa_2;\kappa_3;z\right)=\sum_{n=0}^{\infty}\frac{\mathcal{B}(\kappa_2+n,\kappa_3-\kappa_2)}{\mathcal{B}(\kappa_2,\kappa_3-\kappa_2)}\,\frac{\omega^n}{n!}, \tag{4}$$

$$(|z|<1,\ \Re(\kappa_3)>\Re(\kappa_2)>0).$$

A new extended beta function in terms of extended Mittag-Leffler function given by Khan et al. [7] is defined as follows:

$$\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1,\kappa_2)=\int_0^1\tau^{\kappa_1-1}\left(1-\tau\right)^{\kappa_2-1}E_{\alpha,\beta}\left(-\frac{\wp}{\tau^{\eta}(1-\tau)^{\nu}}\right)d\tau, \tag{5}$$

$$(\Re(\kappa_1)>0,\ \Re(\kappa_2)>0;\alpha,\beta,\eta,\nu\in\mathbb{R}_0^+;\ \wp\geq 0).$$

Here, $E_{\alpha,\beta}(.)$ is known as two-parameter Mittag-Leffler function [13, 14] defined as follows:

$$E_{\alpha,\beta}\left(z\right)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\qquad(\alpha,\beta\in\mathbb{R}_0^+,z\in\mathbb{C}).$$

With the help of new extended beta functions (5), the extended Gauss hypergeometric and confluent hypergeometric functions and their integral representation are defined as follows:

$$F_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1,\kappa_2;\kappa_3;z) = \sum_{n=0}^{\infty}(a)_n \frac{\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_2+n,\kappa_3-\kappa_2)}{\mathcal{B}(\kappa_2,\kappa_3-\kappa_2)}\frac{z^n}{n!}, \tag{6}$$

$$(\wp \geq 0,\ |z|<1,\ \alpha,\beta,\eta,\nu \in \mathbb{R}_0^+,\ \Re(\kappa_3) > \Re(\kappa_2) > 0),$$

$$\Phi_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_2;\kappa_3;z) = \sum_{n=0}^{\infty} \frac{\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_2+n,\kappa_3-\kappa_2)}{\mathcal{B}(\kappa_2,\kappa_3-\kappa_2)}\frac{z^n}{n!}, \tag{7}$$

$$(\wp \geq 0,\ \alpha,\beta,\eta,\nu \in \mathbb{R}_0^+,\ \Re(\kappa_3) > \Re(\kappa_2) > 0).$$

$$\begin{aligned} F_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1,\kappa_2;\kappa_3;z) =& \frac{1}{\mathcal{B}(\kappa_2,\kappa_3-\kappa_2)} \\ &\times \int_0^1 \tau^{\kappa_2-1}(1-\tau)^{\kappa_3-\kappa_2-1}(1-z\tau)^{-\kappa_1} E_{\alpha,\beta}\left(-\frac{\wp}{\tau^{\eta}(1-\tau)^{\nu}}\right)d\tau, \end{aligned} \tag{8}$$

$$(\wp \in \mathbb{R}_0^+, \alpha,\beta,\eta,\nu \in \mathbb{R}_0^+\ and\ |\arg(1-z)| < \pi;\ \Re(\kappa_3) > \Re(\kappa_2) > 0),$$

and

$$\begin{aligned} \Phi_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_2;\kappa_3;z) =& \frac{1}{\mathcal{B}(\kappa_2,\kappa_3-\kappa_2)} \\ &\times \int_0^1 \tau^{\kappa_2-1}(1-\tau)^{\kappa_3-\kappa_2-1} e^{z\tau} E_{\alpha,\beta}\left(-\frac{\wp}{\tau^{\eta}(1-\tau)^{\nu}}\right)d\tau, \end{aligned} \tag{9}$$

$$(\wp \in \mathbb{R}_0^+, \alpha,\beta,\eta,\nu \in \mathbb{R}_0^+\ and\ \Re(\kappa_3) > \Re(\kappa_2) > 0).$$

This research aims to investigate the extended Riemann–Liouville fractional derivative operator, determine the formula of some known functions through the use of extended Appell's functions, and discuss the generating functions of generalized extended hypergeometric functions through the application of a new generalized extended fractional derivative operator. Here is a summary of our paper.

In Sect. 2, we define a generalization of two variables' extended Appell's function $F_{1;\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1,\kappa_2,\kappa_3;\kappa_4;x,y)$, $F_{2;\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1,\kappa_2,\kappa_3;\kappa_4,\kappa_5;x,y)$ and a generalized extended Lauricella's hypergeometric functions of three variables $F_{\mathcal{D},\wp}^{3,\alpha,\beta,\eta,\nu}(\kappa_1,\kappa_2,\kappa_3,\kappa_4;\kappa_5;x,y,z)$. Also, we study integral transforms of generalized extended Appell's function of two variables. In Sect. 3, we define generalized extended Riemann–Liouville fractional derivative operator with the help of extended beta function and find fractional derivative formula of some known functions. In Sect. 4, we introduce the Mellin transform of extended Riemann–Liouville fractional derivative. Finally, in Sect. 5, we derive the generating function of generalized extended hypergeometric functions by using extended fractional derivative operator.

2 An Extended Appell's Functions

Let us define the further extensions of Appell's functions of two variables $F_{1;\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1,\kappa_2,\kappa_3;\kappa_4;x,y)$, $F_{2;\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1,\kappa_2,\kappa_3;\kappa_4,\kappa_5;x,y)$ as well as Lauricella's hypergeometric function of three variables $F_{\mathcal{D},\wp}^{3,\alpha,\beta,\eta,\nu}(\kappa_1,\kappa_2,\kappa_3,\kappa_4;\kappa_5;x,y,z)$ with the help of new extended beta functions (5) as follows:

$$F_{1;\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1,\kappa_2,\kappa_3;\kappa_4;x,y)=\sum_{n,m=0}^{\infty}\frac{\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1+m+n,\kappa_4-\kappa_1)}{\mathcal{B}(\kappa_1,\kappa_4-\kappa_1)}(\kappa_2)_n(\kappa_3)_m\frac{x^n}{n!}\frac{y^m}{m!}, \tag{10}$$

$$(\max\{|x|,|y|\}<1\ ;\ \ \alpha,\beta,\vartheta,\nu\in\mathbb{R}^+\ \ and\ \wp\geq 0).$$

$$F_{2;\alpha,\beta,}^{\wp,\eta,\nu}(\kappa_1,\kappa_2,\kappa_3;\kappa_4,\kappa_5;x,y)$$

$$=\sum_{n,m=0}^{\infty}\frac{(\kappa_1)_{m+n}\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_2+n,\kappa_4-\kappa_2)\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_3+m,\kappa_5-\kappa_3)}{\mathcal{B}(\kappa_2,\kappa_4-\kappa_2)\mathcal{B}(\kappa_3;\kappa_5-\kappa_3)}\frac{x^n}{n!}\frac{y^m}{m!}, \tag{11}$$

$$(|x|+|y|<1;\quad \alpha,\beta,\vartheta,\nu,\in\mathbb{R}^+\ \ and\ \wp\geq 0).$$

$$F_{\mathcal{D},\wp}^{3,\alpha,\beta,\eta,\nu}(\kappa_1,\kappa_2,\kappa_3,\kappa_4;\kappa_5;x,y,z)$$

$$=\sum_{n,m,r=0}^{\infty}\frac{\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1+m+n+r,\kappa_5-\kappa_1)(\kappa_2)_m(\kappa_3)_n(\kappa_4)_r}{\mathcal{B}(\kappa_1,\kappa_5-\kappa_1)}\frac{x^n}{n!}\frac{y^m}{m!}\frac{z^r}{r!}, \tag{12}$$

$$(\sqrt{|x|}+\sqrt{|y|}+\sqrt{|z|}<1\ ;\ \ \alpha,\beta,\vartheta,\nu\in\mathbb{R}_0^+\ \ and\ \wp\geq 0).$$

Remark 1 If $\alpha = \beta = \nu = \eta = 1$ in (10), (11) and (12) then we get the well-known result defined by Özarslan et al. (see [16] Eq. 3, 4, 5).

3 Integral Representation

In this section, we discuss the integral representation of $F_{1;\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1, \kappa_2, \kappa_3; \kappa_4; x, y)$ and $F_{2;\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1, \kappa_2, \kappa_3; \kappa_4, \kappa_5; x, y)$ as follows:

Theorem 1 *The following integral representation of* $F_{1;\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1, \kappa_2, \kappa_3; \kappa_4; x, y)$ *holds true:*

$$F_{1;\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1, \kappa_2, \kappa_3; \kappa_4; x, y)$$

$$= \frac{\Gamma(\kappa_4)}{\Gamma(\kappa_1)\Gamma(\kappa_4 - \kappa_1)} \int_0^1 \tau^{\kappa_1 - 1}(1-\tau)^{\kappa_4 - \kappa_1 - 1}(1 - x\tau)^{-\kappa_2}(1 - y\tau)^{-\kappa_3} E_{\alpha,\beta}\left(\frac{-\wp}{\tau^{\eta}(1-\tau)^{\nu}}\right) d\tau, \tag{13}$$

$$(\wp \geq 0 \ , |arg(1-x)| < \pi, \ |arg(1-y)| < \pi; \ \Re(\kappa_4) > \Re(\kappa_1) > 0)$$

$$(\Re(\kappa_2) > 0, \ \Re(\kappa_3) > 0; \ \eta, \nu, \alpha, \beta \in \mathbb{R}_0^+).$$

Proof Let us assume that if $|x| < 1, \ |y| < 1, \ \Re(\kappa_2) > 0$ *and* $\Re(\kappa_3) > 0$. By using the binomial expansion of $(1 - x\tau)^{-\kappa_2}$ and $(1 - y\tau)^{-\kappa_3}$ and considering the fact that the series involved are uniformly convergent and the given integral

$$\int_0^1 \tau^{\kappa_1 + m + n - 1}(1-\tau)^{\kappa_4 - \kappa_1 - 1} E_{\alpha,\beta}\left(\frac{-\wp}{\tau^{\eta}(1-\tau)^{\nu}}\right) d\tau,$$

is absolutely convergent for $m, n \in \mathbb{N}_0, \Re(\kappa_4) > \Re(\kappa_1) > 0; \quad \eta, \nu, \alpha, \beta \in \mathbb{R}^+$ and $\wp \geq 0$ because of the fact that

$$\left| \int_0^1 \tau^{\kappa_1 + m + n - 1}(1-\tau)^{\kappa_4 - \kappa_1 - 1} E_{\alpha,\beta}\left(\frac{-\wp}{\tau^{\eta}(1-\tau)^{\nu}}\right) d\tau \right| \leq \int_0^1 \tau^{\kappa_1 + m + n - 1}(1-\tau)^{\kappa_4 - \kappa_1 - 1} d\tau,$$

we have by interchanging the order of summation and integration that

$$\int_0^1 \tau^{\kappa_1-1}(1-\tau)^{\kappa_4-\kappa_1-1}(1-x\tau)^{-\kappa_2}(1-y\tau)^{-\kappa_3}E_{\alpha,\beta}\left(\frac{-\wp}{\tau^{\eta}(1-\tau)^{\nu}}\right)d\tau$$

$$=\int_0^1 \tau^{\kappa_1-1}(1-\tau)^{\kappa_4-\kappa_1-1}E_{\alpha,\beta}\left(\frac{-\wp}{\tau^{\eta}(1-\tau)^{\nu}}\right)\sum_{n=0}^{\infty}(\kappa_2)_n\frac{(x\tau)^n}{n!}\sum_{m=0}^{\infty}(\kappa_3)_m\frac{(y\tau)^m}{m!}d\tau$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(\kappa_2)_n(\kappa_3)_m\frac{x^n}{n!}\frac{y^m}{m!}\int_0^1 \tau^{\kappa_1+m+n-1}(1-\tau)^{\kappa_4-\kappa_1-1}E_{\alpha,\beta}\left(\frac{-\wp}{\tau^{\eta}(1-\tau)^{\nu}}\right)d\tau$$

$$=\sum_{n,m=0}^{\infty}\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1+m+n,\kappa_4-\kappa_1)(\kappa_2)_n(\kappa_3)_m\frac{x^n}{n!}\frac{y^m}{m!}$$

$$=\mathcal{B}(\kappa_1,\kappa_4-\kappa_1)\sum_{n,m=0}^{\infty}\frac{\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1+m+n,\kappa_4-\kappa_1)}{\mathcal{B}(\kappa_1,\kappa_4-\kappa_1)}(\kappa_2)_n(\kappa_3)_m\frac{x^n}{n!}\frac{y^m}{m!}$$

$$=\frac{\Gamma(\kappa_1)\Gamma(\kappa_4-\kappa_1)}{\Gamma(\kappa_4)}F_{1;\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1,\kappa_2,\kappa_3;\kappa_4;x,y).$$

□

Theorem 2 *The following integral representation of* $F_{2;\alpha,\beta}^{p,\eta,\nu}(\kappa_1,\kappa_2,\kappa_3;\kappa_4,\kappa_5;x,y)$ *holds true:*

$$F_{2;\alpha,\beta,}^{\wp,\eta,\nu,}(\kappa_1,\kappa_2,\kappa_3;\kappa_4,\kappa_5;x,y)=\frac{1}{\mathcal{B}(\kappa_2,\kappa_4-\kappa_2)\mathcal{B}(\kappa_3,\kappa_5-\kappa_3)}\times\int_0^1\int_0^1\frac{\tau^{\kappa_2-1}(1-\tau)^{\kappa_4-\kappa_2-1}s^{\kappa_3-1}(1-s)^{\kappa_5-\kappa_3-1}}{(1-x\tau-ys)^{\kappa_1}}\times E_{\alpha,\beta}\left(\frac{-\wp}{s^{\eta}(1-s)^{\nu}}\right)E_{\alpha,\beta}\left(\frac{-\wp}{s^{\eta}(1-s)^{\nu}}\right)d\tau\,ds, \quad (14)$$

$(\wp\geq 0;\ |x|+|y|<1;\ \Re(\kappa_4)>\Re(\kappa_2)>0;\ \Re(\kappa_5)>\Re(\kappa_3)>0;\ \Re(\kappa_1)>0;\ \alpha,\beta,\eta,\nu\in\mathbb{R}^+)$.

Proof Suppose that $|x|+|y|<1$ and $\Re(\kappa_1)>0$ expanding $(1-x\tau-ys)^{\kappa_1}$, we get

$$\int_0^1\int_0^1\frac{\tau^{\kappa_2-1}(1-\tau)^{\kappa_4-\kappa_2-1}s^{\kappa_3-1}(1-s)^{\kappa_5-\kappa_3-1}}{(1-x\tau-ys)^{\kappa_1}}E_{\alpha,\beta}\left(\frac{-\wp}{s^{\eta}(1-s)^{\nu}}\right)E_{\alpha,\beta}\left(\frac{-\wp}{s^{\eta}(1-s)^{\nu}}\right)d\tau\,ds$$

$$= \int_0^1 \int_0^1 \tau^{\kappa_2-1}(1-\tau)^{\kappa_4-\kappa_2-1} E_{\alpha,\beta}\left(\frac{-\wp}{s^{\eta}(1-s)^{\nu}}\right) s^{\kappa_3-1}(1-s)^{\kappa_5-\kappa_3-1} E_{\alpha,\beta}\left(\frac{-\wp}{s^{\eta}(1-s)^{\nu}}\right)$$
$$\times \sum_{N=0}^{\infty} \frac{(\kappa_1)_N (x\tau + ys)^N}{N!} d\tau\, ds,$$

we know that the summation formula,

$$\left\{ \sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(m+n) \frac{x^n y^m}{n!m!} \right\}.$$

We obtain

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (\kappa_1)_{m+n} \int_0^1 \int_0^1 \tau^{\kappa_2-1}(1-\tau)^{\kappa_4-\kappa_2-1} E_{\alpha,\beta}\left(\frac{-\wp}{s^{\eta}(1-s)^{\nu}}\right)$$
$$\times s^{\kappa_3-1}(1-s)^{\kappa_5-\kappa_3-1} E_{\alpha,\beta}\left(\frac{-\wp}{s^{\eta}(1-s)^{\nu}}\right) \frac{(x\tau)^n}{n!} \frac{(ys)^m}{m!} dtds$$
$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (\kappa_1)_{m+n} \int_0^1 \tau^{\kappa_2+n-1}(1-\tau)^{\kappa_4-\kappa_2-1} E_{\alpha,\beta}\left(\frac{-\wp}{s^{\eta}(1-s)^{\nu}}\right) d\tau$$
$$\times \int_0^1 s^{\kappa_3+m-1}(1-s)^{\kappa_5-\kappa_3-1} E_{\alpha,\beta}\left(\frac{-\wp}{s^{\eta}(1-s)^{\nu}}\right) ds\, \frac{x^n y^m}{n!m!}$$
$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (\kappa_1)_{m+n} \mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_2+n, \kappa_4-\kappa_2) \mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_3+m, \kappa_5-\kappa_3) \frac{x^n y^m}{n!m!}$$
$$= \mathcal{B}(\kappa_2, \kappa_4-\kappa_2)\mathcal{B}(\kappa_3, \kappa_5-\kappa_3) F_{2;\alpha,\beta}^{\wp,\eta,\nu,}(\kappa_1, \kappa_2, \kappa_3; \kappa_4, \kappa_5; x, y).$$

Hence, we get the desired result (14). □

4 An Extended Riemann–Liouville Fractional Derivative

The well-known classical Riemann–Liouville fractional derivative of order ϑ defined as

$$\mathcal{D}_z^{\vartheta}\{f(z)\} = \frac{1}{\Gamma(-\vartheta)} \int_0^z f(\tau)(z-\tau)^{-\vartheta-1} d\tau, \quad (\Re(\vartheta) < 0),$$

where the line on complex τ plane that goes from 0 to z and for $j-1 < \Re(\vartheta) < j$ such that $j = 1, 2, 3, ...$, we have

$$\mathcal{D}_z^{\vartheta}\{f(z)\} = \frac{d^j}{dz^j}\mathcal{D}_z^{\vartheta-j}\{f(z)\}$$
$$= \frac{d^j}{dz^j}\left\{\frac{1}{\Gamma(j-\vartheta)}\int_0^z f(\tau)(z-1)^{-\vartheta+j-1}d\tau\right\}.$$

The study of fractional calculus has become a popular research subject due to its numerous applications in diverse fields of science and engineering, including fluid flow, electrical networks, and probability theory. Srivastava et al. [24] and several authors [3, 5] provide a well-organized summary of the research on the topic of fractional calculus and its applications, and this is a valuable contribution. Srivastava and Manocha [25] also provide an explanation for the application of fractional derivatives in generating functions theory.

After introducing new parameters, we proposed an extended Riemann–Liouville fractional derivative operator using the Mittag-Leffler function as a kernel operator defined as follows:

$$\mathcal{D}_{z;\alpha,\beta}^{\vartheta,p,\eta,\nu}\{f(z)\} = \frac{1}{\Gamma(-\vartheta)}\int_0^z f(\tau)(z-\tau)^{-\vartheta-1}E_{\alpha,\beta}\left(\frac{-\wp z^{\eta+\nu}}{\tau^n(z-\tau)^\nu}\right)d\tau, \tag{15}$$

$$(\Re(\vartheta) < 0; \Re(p) > 0; p = 0; \eta, \nu, \alpha, \beta \in \mathbb{R}_0^+),$$

and for $j-1 < \Re(\vartheta) < j$ such that $j = 1, 2, 3, ...$

Here, an extended Riemann–Liouville fractional derivative operator $\mathcal{D}_{z;\alpha,\beta}^{\vartheta,p,\eta,\nu}$ has an important semi-group property (or convolution property), i.e., $\mathcal{D}_{z;\alpha,\beta}^{\vartheta,p,\eta,\nu} * \mathcal{D}_{z;\alpha,\beta}^{\zeta,p,\eta,\nu} = \mathcal{D}_{z;\alpha,\beta}^{\vartheta+\zeta,p,\eta,\nu}$ for arbitrary $\Re(\vartheta) < 0$ and $\Re(\zeta) < 0$.

$$\mathcal{D}_{z;\alpha,\beta}^{\vartheta,p,\eta,\nu}\{f(z)\} = \frac{d^j}{dz^j}\mathcal{D}_z^{\vartheta-j}\{f(z)\}$$
$$= \frac{d^j}{dz^j}\left\{\frac{1}{\Gamma(j-\vartheta)}\int_0^z f(\tau)(z-1)^{-\vartheta+j-1}E_{\alpha,\beta}\left(\frac{-\wp z^{\eta+\nu}}{\tau^n(z-\tau)^\nu}\right)d\tau\right\},$$

where the integration path is a line in the complex t-plane from 0 to z.

Remark 2 The classical Riemann–Liouville fractional derivative operator is obtained when $\wp = \alpha = \beta = \vartheta = \eta = 0$ and if $\alpha = \beta = \vartheta = \eta = 1$ in (15) then we obtained the known result (see [16], Eq. 6).

5 Applications

The generalized Riemann–Liouville fractional derivative operator has been used to compute the fractional derivative formulae of a few fundamental functions in the following theorems.

Theorem 3 *The following relation holds true:*

$$\mathcal{D}^{\vartheta,\wp,\eta,\nu}_{z,\alpha,\beta}(z^{\varrho}) = \frac{z^{\varrho-\vartheta}}{\Gamma(-\vartheta)}\mathcal{B}^{\wp,\eta,\nu}_{\alpha,\beta}(\varrho+1,-\vartheta), \tag{16}$$

$$(\Re(\varrho) > -1;\ \Re(\vartheta) < 0;\ \alpha,\beta,\eta,\nu \in \mathbb{R}^+;\ \wp \geq 0).$$

Proof By using (15), we have

$$\mathcal{D}^{\vartheta,\wp,\eta,\nu}_{z,\alpha,\beta}(z^{\varrho}) = \frac{1}{\Gamma(-\vartheta)}\int_0^z \tau^{\varrho}(z-\tau)^{-\vartheta-1}E_{\alpha,\beta}\left(\frac{-\wp z^{\eta+\nu}}{\tau^{\eta}(z-\tau)^{\nu}}\right)d\tau,$$

by putting $\tau = zv, d\tau = zdv$ in above equation, we get

$$\begin{aligned}
&=\frac{1}{\Gamma(-\vartheta)}\int_0^z (zv)^{\varrho}(z-zv)^{-\varrho-1}E_{\alpha,\beta}\left(\frac{-\wp z^{\eta+\nu}}{z^{\eta+\nu}v^{n}(1-v)^{\nu}}\right)zdv\\
&=\frac{z^{\varrho+\vartheta}}{\Gamma(-\vartheta)}\int_0^1 v^{\varrho}(1-v)^{-\vartheta-1}E_{\alpha,\beta}\left(\frac{-p}{v^{\eta}(1-v)^{\nu}}\right)du\\
&=\frac{z^{\varrho-\vartheta}}{\Gamma(-\vartheta)}\mathcal{B}^{\wp,\eta,\nu}_{\alpha,\beta}(\varrho+1,-\vartheta).
\end{aligned}$$

Thus, we obtain what we want. □

Theorem 4 *For* $\Re(\varrho) > 0$; $\Re(\kappa_1) > 0$; $\Re(\vartheta) < 0$ *and* $|z| < 1$, *the following relation holds true:*

$$\mathcal{D}^{\varrho-\vartheta,\wp,\eta,\nu}_{z;\alpha,\beta}\left\{z^{\varrho-1}(1-z)^{-\kappa_1}\right\} = \frac{\Gamma(\varrho)}{\Gamma(\vartheta)}z^{\vartheta-1}F^{\wp,\eta,\nu}_{\alpha,\beta}(\kappa_1,\varrho;\vartheta;z). \tag{17}$$

Proof Using (15), we have

$$\mathcal{D}^{\varrho-\vartheta,\wp,\eta,\nu}_{z;\alpha,\beta}\left\{z^{\varrho-1}(1-z)^{-\kappa_1}\right\}$$

$$=\frac{1}{\Gamma(\vartheta-\varrho)}\int_0^z \tau^{\varrho-1}(1-\tau)^{-\kappa_1}(z-\tau)^{\vartheta-\varrho-1}E_{\alpha,\beta}\left(\frac{-\wp z^{\eta+\nu}}{\tau^{\eta}(z-\tau)^{\nu}}\right)d\tau,$$

$$=\frac{z^{\vartheta-\varrho-1}}{\Gamma(\vartheta-\varrho)}\int_0^z \tau^{\varrho-1}(1-\tau)^{-\kappa_1}(1-\frac{\tau}{z})^{\vartheta-\varrho-1}E_{\alpha,\beta}\left(\frac{-\wp z^{\eta+\nu}}{\tau^{n}(z-\tau)^{\nu}}\right)d\tau.$$

By inserting $\tau = zv,\ d\tau = zdv$ in the above equation, we have

$$=\frac{z^{\vartheta-\varrho-1}}{\Gamma(\vartheta-\varrho)}\int_0^1 (zv)^{\varrho-1}(1-zv)^{-\kappa_1}(1-v)^{\vartheta-\varrho-1}E_{\alpha,\beta}\left(\frac{-\wp z^{\eta+\nu}}{z^{\eta+\nu}v^{\eta}(1-v)^{\nu}}\right)zdv,$$

$$=\frac{z^{\vartheta-1}}{\Gamma(\vartheta-\varrho)}\int_0^1 v^{\varrho-1}(1-v)^{\vartheta-\varrho-1}(1-zv)^{-\kappa_1}E_{\alpha,\beta}\left(\frac{-p}{v^{\eta}(1-v)^{\nu}}\right)dv.$$

Using (8) in the above expression, we get

$$\begin{aligned}\mathcal{D}_{z;\alpha,\beta}^{\varrho-\vartheta,\wp,\eta,\nu}\left\{z^{\varrho-1}(1-z)^{-\kappa_1}\right\} &=\frac{z^{\vartheta-1}}{\Gamma(\vartheta-\varrho)}\mathcal{B}(\varrho,\vartheta-\varrho)F_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1,\varrho;\vartheta;z)\\ &=\frac{z^{\vartheta-1}}{\Gamma(\vartheta-\varrho)}\frac{\Gamma(\varrho)\Gamma(\vartheta-\varrho)}{\Gamma(\vartheta)}F_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1,\varrho;\vartheta;z)\\ &=\frac{\Gamma(\varrho)}{\Gamma(\vartheta)}z^{\vartheta-1}F_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1,\varrho;\vartheta;z).\end{aligned}$$

Thus, we arrive at the desired result. □

Theorem 5 *For* $\Re(\varrho) > \Re(\vartheta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $|\kappa_1 z| < 1$, *and* $|\kappa_2 z| < 1$, *the following relation holds true:*

$$\mathcal{D}_{z;\alpha,\beta}^{\varrho-\vartheta,\wp,\eta,\nu}\left\{z^{\varrho-1}(1-\kappa_1 z)^{-\gamma}(1-\kappa_2 z)^{-\delta}\right\} = z^{\vartheta-1}\frac{\Gamma(\varrho)}{\Gamma(\vartheta)}F_{1;\alpha,\beta}^{\wp,\eta,\nu}(\varrho,\gamma,\delta,\epsilon;\vartheta;\kappa_1 z,\kappa_2 z). \quad (18)$$

Proof By using (15), we have

$$\mathcal{D}_{z;\alpha,\beta}^{\varrho-\vartheta,\wp,\eta,\nu}\left\{z^{\varrho-1}(1-\kappa_1 z)^{-\gamma}(1-\kappa_2 z)^{-\delta}\right\}$$

$$=\frac{1}{\Gamma(\vartheta-\varrho)}\int_0^z \tau^{\varrho-1}(1-\kappa_1\tau)^{-\gamma}(1-\kappa_2\tau)^{-\delta}E_{\alpha,\beta}\left(\frac{-\wp z^{\eta+\nu}}{\tau^{n}(z-\tau)^{\nu}}\right)(z-\tau)^{\vartheta-\varrho-1}d\tau,$$

$$=\frac{z^{\vartheta-\varrho-1}}{\Gamma(\vartheta-\varrho)}\int_0^z \tau^{\varrho-1}(1-\kappa_1\tau)^{-\gamma}(1-\kappa_2\tau)^{-\delta}(1-\frac{\tau}{z})^{\vartheta-\varrho-1}E_{\alpha,\beta}\left(\frac{-\wp z^{\eta+\nu}}{\tau^{\eta}(z-\tau)^{\nu}}\right)d\tau.$$

Putting $\tau = zv, d\tau = zdv$,

$$=\frac{z^{\vartheta-\varrho-1}}{\Gamma(\vartheta-\varrho)}\int_0^z (zv)^{\varrho-1}(1-\kappa_1 vz)^{-\gamma}(1-\kappa_2 vz)^{-\delta}(1-v)^{\vartheta-\varrho-1}E_{\alpha,\beta}\left(\frac{-\wp z^{\eta+\nu}}{(zv)^{\eta}(1-zv)^{\nu}}\right)zdv$$
$$=\frac{z^{\vartheta-1}}{\Gamma(\vartheta-\varrho)}\int_0^z (v)^{\varrho-1}(1-\kappa_1 vz)^{-\gamma}(1-\kappa_2 vz)^{-\delta}(1-v)^{\vartheta-\varrho-1}E_{\alpha,\beta}\left(\frac{-\wp}{v^{\eta}(1-v)^{\nu}}\right)dv.$$

Applying definition (13) in the above expression, we get

$$\mathcal{D}^{\varrho-\vartheta,\wp,\eta,\nu}_{z;\alpha,\beta}\left\{z^{\varrho-1}(1-\kappa_1 z)^{-\gamma}(1-\kappa_2 z)^{-\delta}\right\} =\frac{z^{\vartheta-1}}{\Gamma(\vartheta-\varrho)}\frac{\Gamma(\varrho)\Gamma(\vartheta-\varrho)}{\Gamma(\vartheta)}$$
$$\times F^{\wp,\eta,\nu}_{1;\alpha,\beta}(\varrho,\gamma,\delta;\vartheta;\kappa_1 z,\kappa_2 z)$$
$$=z^{\vartheta-1}\frac{\Gamma(\varrho)}{\Gamma(\vartheta)}F^{\wp,\eta,\nu}_{1;\alpha,\beta}(\varrho,\gamma,\delta;\vartheta;\kappa_1 z,\kappa_2 z).$$

We complete the proof of this theorem. □

Theorem 6 *For* $\Re(\varrho) > \Re(\vartheta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\epsilon) > 0$, $|\kappa_1 z| < 1$, $|\kappa_2 z| < 1$, *and* $|\kappa_3 z| < 1$, *the following relation holds true:*

$$\mathcal{D}^{\varrho-\vartheta,\wp,\eta,\nu}_{z;\alpha,\beta}\left\{z^{\varrho-1}(1-\kappa_1 z)^{-\gamma}(1-\kappa_2 z)^{-\delta}(1-\kappa_3 z)^{-\epsilon}\right\}$$
$$=z^{\vartheta-1}\frac{\Gamma(\varrho)}{\Gamma(\vartheta)}F^{3,\eta,\nu}_{D,p,\alpha,\beta}(\varrho,\gamma,\delta,\epsilon;\vartheta;\kappa_1 z,\kappa_2 z,\kappa_3 z), \tag{19}$$

$$(\alpha,\beta,\eta,\nu\in\mathbb{R}^+ \ \ and \ \ \wp\geq 0).$$

Proof With the help of (18), we get

$$\mathcal{D}^{\varrho-\vartheta,\wp,\eta,\nu}_{z;\alpha,\beta}\left\{z^{\varrho-1}(1-\kappa_1 z)^{-\gamma}(1-\kappa_2 z)^{-\delta}(1-\kappa_3 z)^{-\epsilon}\right\}$$

$$=\frac{z^{\vartheta-1}}{\Gamma(\vartheta-\varrho)}\sum_{m,n,r=0}^{\infty}\frac{(\gamma)_m(\delta)_n(\epsilon)_r}{m!n!r!}\kappa_1^m\ \kappa_2^n\ \kappa_3^r\ \mathcal{B}^{\wp,\eta,\nu}_{\alpha,\beta}(\varrho+m+n+r,\vartheta-\varrho)z^{m+n+r}$$
$$=\frac{\mathcal{B}(\varrho,\vartheta-\varrho)}{\Gamma(\vartheta-\varrho)}z^{\vartheta-1}\sum_{m,n,r=0}^{\infty}\frac{\mathcal{B}^{\wp,\eta,\nu}_{\alpha,\beta}(\varrho+m+n+r,\vartheta-\varrho)}{\mathcal{B}(\varrho,\vartheta-\varrho)}\frac{(\gamma)_m(\delta)_n(\epsilon)_r}{m!n!r!}$$
$$\times\kappa_1^m\kappa_2^n\kappa_3^r(\kappa_1 z)^m(\kappa_2 z)^n(\kappa_3 z)^r$$
$$=z^{\vartheta-1}\frac{\Gamma(\varrho)}{\Gamma(\vartheta)}F^{3,\eta,\nu}_{D,\wp,\alpha,\beta}(\varrho,\gamma,\delta,\epsilon;\vartheta;\kappa_1 z,\kappa_2 z,\kappa_3 z).$$

Thus, we obtain what we want. □

Theorem 7 *For* $\Re(\varrho) > \Re(\vartheta) > 0$, $\Re(\kappa_1) > 0$, $\Re(\kappa_2) > 0$, $\Re(\gamma) > 0$, *and* $\left|\frac{x}{1-z}\right| < 1$, *the following relation holds true:*

$$\mathcal{D}_{z;\alpha,\beta}^{\varrho-\vartheta,\wp,\eta,\nu}\left\{z^{\varrho-1}(1-z)^{-\kappa_1}F_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1,\kappa_2;\gamma;\frac{x}{1-z})\right\}$$

$$=\frac{1}{\mathcal{B}(\kappa_2,\gamma-\kappa_2)\Gamma(\vartheta-\varrho)}z^{\vartheta-1}F_{2;\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1,\kappa_2,\varrho;\gamma,\vartheta;x,z), \tag{20}$$

$$(\alpha,\beta,\eta,\nu\in\mathbb{R}^+ \ \ and \ \ \wp\geq 0).$$

Proof By using the relation (15) and (11), we obtain

$$\mathcal{D}_{z;\alpha,\beta}^{\varrho-\vartheta,\wp,\eta,\nu}\left\{z^{\varrho-1}(1-z)^{-\kappa_1}F_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1,\kappa_2;\gamma;\frac{x}{1-z})\right\}$$

$$=\mathcal{D}_{z;\alpha,\beta}^{\varrho-\vartheta,\wp,\eta,\nu}\left\{z^{\varrho-1}(1-z)^{-\kappa_1}\frac{1}{\mathcal{B}(\kappa_2,\gamma-\kappa_2)}\sum_{n=0}^{\infty}\frac{(\kappa_1)_n\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_2+n,\gamma-\kappa_2)}{n!}\left(\frac{x}{1-z}\right)^n\right\}$$

$$=\frac{1}{\mathcal{B}(b,\gamma-b)}D_{z;\alpha,\beta}^{\varrho-\vartheta,\wp,\eta,\nu}\left\{z^{\varrho-1}\sum_{n=0}^{\infty}(\kappa_1)_n\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_2+n,\gamma-\kappa_2)\frac{x^n}{n!}(1-z)^{-\kappa_1-n}\right\}$$

$$=\frac{1}{\mathcal{B}(\kappa_2,\gamma-\kappa_2)}\sum_{n=0}^{\infty}(\kappa_1)_n\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_2+n,\gamma-\kappa_2)\frac{x^n}{n!}D_{z;\alpha,\beta}^{\varrho-\vartheta,p,\eta,\nu}\left\{z^{\varrho-1}(1-z)^{-\kappa_1-n}\right\}$$

$$=\frac{1}{\mathcal{B}(\kappa_2,\gamma-\kappa_2)}\sum_{n=0}^{\infty}(\kappa_1)_n\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_2+n,\gamma-\kappa_2)\frac{x^n}{n!}\frac{z^{\vartheta-1}}{\Gamma(\vartheta-\varrho)}\mathcal{B}(\varrho,\vartheta-\varrho)$$
$$\times F_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1+n,\varrho;\vartheta;z)$$

$$=\frac{z^{\vartheta-1}}{\mathcal{B}(\kappa_2,\gamma-\kappa_2)\Gamma(\vartheta-\varrho)}\sum_{n=0}^{\infty}(\kappa_1)_n\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_2+n,\gamma-\kappa_2)\frac{x^n}{n!}\mathcal{B}(\varrho,\vartheta-\varrho)$$
$$\times F_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1+n,\varrho;\vartheta;z)$$

$$=\frac{z^{\vartheta-1}}{\mathcal{B}(\kappa_2,\gamma-\kappa_2)\Gamma(\vartheta-\varrho)}$$
$$\times\sum_{n=0}^{\infty}(\kappa_1)_n\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_2+n,\gamma-\kappa_2)\frac{x^n}{n!}\mathcal{B}(\varrho,\vartheta-\varrho)\sum_{m=0}^{\infty}(\kappa_1+n)_m\frac{\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\varrho+m,\vartheta-\varrho)}{\mathcal{B}(\varrho,\vartheta-\varrho)}\frac{z^m}{m!}$$

$$=\frac{z^{\vartheta-1}}{\mathcal{B}(\kappa_2,\gamma-\kappa_2)\Gamma(\vartheta-\varrho)}\sum_{n,m=0}^{\infty}(\kappa_1)_n(\kappa_1+n)_m\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_2+n,\gamma-\kappa_2)\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\varrho+m,\vartheta-\varrho)$$
$$\times\frac{x^nz^m}{n!m!}$$

$$=\frac{z^{\vartheta-1}}{\mathcal{B}(\kappa_2,\gamma-\kappa_2)\Gamma(\vartheta-\varrho)}\sum_{n,m=0}^{\infty}(\kappa_1)_{n+m}\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_2+n,\gamma-\kappa_2)\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\varrho+m,\vartheta-\varrho)\frac{x^nz^m}{n!m!}$$

$$=\frac{1}{\mathcal{B}(\kappa_2,\gamma-\kappa_2)}\sum_{n,m=0}^{\infty}\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\kappa_2+n,\gamma-\kappa_2)\frac{x^n}{n!}\frac{(\kappa_1)_{n+m}}{m!}\frac{\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\varrho+m,\vartheta-\varrho)}{\Gamma(\vartheta-\varrho)}z^{\vartheta+m-1}$$
$$=\frac{1}{\mathcal{B}(\kappa_2,\gamma-\kappa_2)\Gamma(\vartheta-\varrho)}z^{\vartheta-1}F_{2;\alpha,\beta}^{\wp,\eta,\nu}(\kappa_1,\kappa_2,\varrho;\gamma,\vartheta;x,z).$$

□

Thus, we obtain what we want.

Example 1 Let $f(z)=\sum_{n=0}^{\infty}a_n z^n$ be a power series expansion of an analytic function f(z) in the disk $|z|<\rho$, then following relation holds:

$$\mathcal{D}_{z;\alpha,\beta}^{\vartheta,\wp,\eta,\nu}\left\{z^{\varrho-1}f(z)\right\}=\mathcal{D}_{z;\alpha,\beta}^{\vartheta,\wp,\eta,\nu}\left\{z^{\varrho-1}\sum_{n=0}^{\infty}a_n z^n\right\}$$
$$=\frac{z^{\varrho-\vartheta-1}}{\Gamma(-\vartheta)}\sum_{n=0}^{\infty}a_n\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\varrho+n,-\vartheta)z^n,$$

$$(\Re(\varrho)>0,\ \ \Re(\vartheta)<0,\ \ \alpha,\beta,\eta,\nu\in\mathbb{R}^+\ \ and\ \ \wp\geq 0).$$

Proof We have

$$\mathcal{D}_{z;\alpha,\beta}^{\vartheta,\wp,\eta,\nu}\left\{z^{\varrho-1}f(z)\right\}=\mathcal{D}_{z;\alpha,\beta}^{\vartheta,\wp,\eta,\nu}\left\{z^{\varrho-1}\sum_{n=0}^{\infty}a_n z^n\right\}$$
$$=\sum_{n=0}^{\infty}a_n\mathcal{D}_{z;\alpha,\beta}^{\vartheta,\wp,\eta,\nu}\left\{z^{\varrho+n-1}\right\}$$
$$=\frac{z^{\varrho-\vartheta-1}}{\Gamma(-\vartheta)}\sum_{n=0}^{\infty}a_n\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\varrho+n,-\vartheta)z^n$$
$$=\frac{1}{\Gamma(-\vartheta)}\int_0^z t^{\varrho-1}\sum_{n=0}^{\infty}a_n\tau^{\eta}(z-\tau)^{-\vartheta-1}E_{\alpha,\beta}\left(\frac{-\wp z^{\eta+\nu}}{\tau^{\eta}(z-\tau)^{\nu}}\right)d\tau$$
$$=\frac{z^{-\vartheta-1}}{\Gamma(-\vartheta)}\int_0^z \tau^{\varrho-1}\sum_{n=0}^{\infty}a_n\tau^{\eta}(1-\frac{\tau}{z})^{-\vartheta-1}E_{\alpha,\beta}\left(\frac{-\wp}{(\frac{\tau}{z})^{\eta}(1-\frac{\tau}{z})^{\nu}}\right)d\tau,$$

if we choose $\frac{\tau}{z}=u$ then $zdu=d\tau$, putting this value in above expression, we have

$$=\frac{z^{\vartheta-1}}{\Gamma(-\vartheta)}\int_0^1 (uz)^{\varrho-1}\sum_{n=0}^{\infty}a_n(uz)^n(1-u)^{-\vartheta-1}E_{\alpha,\beta}\left(\frac{-\wp}{u^{\eta}(1-u)^{\nu}}\right)zdu$$

$$=\frac{z^{\varrho-\vartheta-1}}{\Gamma(-\vartheta)}\int_0^1 (u)^{\varrho-1}(1-u)^{-\vartheta-1}E_{\alpha,\beta}\left(\frac{-\wp}{u^{\eta}(1-u)^{\nu}}\right)\sum_{n=0}^{\infty}a_n(uz)^n du.$$

By using uniformly convergent condition

$$=\sum_{n=0}^{\infty}a_n\frac{z^{\varrho+n-\vartheta-1}}{\Gamma(-\vartheta)}\int_0^1 u^{\varrho+n-1}(1-u)^{-\vartheta-1}E_{\alpha,\beta}\left(\frac{-\wp}{u^{\eta}(1-u)^{\nu}}\right)du$$

$$=\sum_{n=0}^{\infty}a_n\frac{z^{\varrho+n-\vartheta-1}}{\Gamma(-\vartheta)}\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\varrho+n,-\vartheta)$$

$$=\frac{z^{\varrho-\vartheta-1}}{\Gamma(-\vartheta)}\sum_{n=0}^{\infty}a_n\mathcal{B}_{\alpha,\beta}^{\wp,\eta,\nu}(\varrho+n,-\vartheta)z^n.$$

Hence proved. □

6 Mellin Transforms

This section deals with significant Mellin transform and extended Riemann–Liouville fractional derivative operator, which have been defined as follows:

Theorem 8 *The Mellin transform of an extended Riemann–Liouville fractional derivative operator is defined in* (15) *as follows:*

$$\mathcal{M}\left[\mathcal{D}_{z,\alpha,\beta}^{\vartheta,\wp,\eta,\nu}\{z^{\varrho}:s\}\right]=\frac{\Gamma_{(s)}^{\alpha,\beta}}{\Gamma(-\vartheta)}\mathcal{B}(\varrho+\eta s+1,\nu s-\vartheta)z^{\varrho-\vartheta}, \tag{21}$$

$$(\Re(\varrho)>-1,\ \ \Re(\vartheta)<0,\ \ \Re(s)>0,\ \ \alpha,\beta,\eta,\nu\in\mathbb{R}^{+}\ \ and\ \ \wp\geq 0).$$

Proof By using definition of Mellin transform and (15), we have

$$\mathcal{M}\left[\mathcal{D}_{z,\alpha,\beta}^{\vartheta,\wp,\eta,\nu}\{z^{\varrho}:s\}\right]$$

$$=\int_0^\infty \mathcal{P}^{s-1}\mathcal{D}_{z;\alpha,\beta}^{\vartheta,\wp,\eta,\nu}(z^\varrho)d\mathcal{P}$$

$$=\frac{1}{\Gamma(-\vartheta)}\int_0^\infty \mathcal{P}^{s-1}\int_0^z \tau^\varrho(z-\tau)^{-\vartheta-1}E_{\alpha,\beta}\left(\frac{-\wp z^{\eta+\nu}}{\tau^\eta(z-\tau)^\nu}\right)d\tau d\mathcal{P}$$

$$=\frac{z^{-\vartheta-1}}{\Gamma(-\vartheta)}\int_0^\infty \mathcal{P}^{s-1}\int_0^z \tau^\varrho(1-\frac{\tau}{z})^{-\vartheta-1}E_{\alpha,\beta}\left(\frac{-\wp z^{\eta+\nu}}{\tau^\eta(z-\tau)^\nu}\right)d\tau d\mathcal{P}$$

$$=\frac{z^{-\vartheta-1}}{\Gamma(-\vartheta)}\int_0^\infty \mathcal{P}^{s-1}\int_0^1 u^\varrho z^\varrho(1-u)^{-\vartheta-1}E_{\alpha,\beta}\left(\frac{-\wp}{u^\eta(1-u)^\nu}\right)z\,du\,d\mathcal{P},$$

since

$$\int_0^1 u^\varrho(1-u)^{-\vartheta-1}E_{\alpha,\beta}\left(\frac{-\wp}{u^\eta(1-u)^\nu}\right)du,$$

and

$$\int_0^\infty \mathcal{P}^{s-1}E_{\alpha,\beta}\left(\frac{-\wp}{u^\eta(1-u)^\nu}\right)d\mathcal{P},$$

are absolutely convergent, the order of integration will be interchanged and we have

$$=\frac{z^{-\vartheta-1}}{\Gamma(-\vartheta)}\int_0^1 u^\varrho(1-u)^{-\vartheta-1}\int_0^\infty \mathcal{P}^{s-1}E_{\alpha,\beta}\left(\frac{-\wp}{u^\eta(1-u)^\nu}\right)d\mathcal{P}du.$$

If we put $\gamma=\frac{\mathcal{P}}{u^\eta(1-u)^\nu}$ implies $u^\eta(1-u)^\nu d\gamma=d\mathcal{P}$, then we see that

$$=\frac{z^{-\vartheta-1}}{\Gamma(-\vartheta)}\int_0^1 u^\varrho(1-u)^{-\vartheta-1}\int_0^\infty (\gamma u^\eta(1-u)^\nu)^{s-1}E_{\alpha,\beta}(-\gamma)u^\eta(1-u)^\nu d\gamma du$$

$$=\frac{z^{\varrho-\vartheta}}{\Gamma(-\vartheta)}\int_0^1 u^{\varrho+\eta s}(1-u)^{\nu s-\vartheta-1}\int_0^\infty \gamma^{s-1}E_{\alpha,\beta}(-\gamma)d\gamma du$$

$$=\frac{\Gamma_{(s)}^{\alpha,\beta}}{\Gamma(-\vartheta)}z^{\varrho-\vartheta}\int_0^1 u^{\varrho+\eta s}(1-u)^{\nu s-\vartheta-1}du$$

$$=\frac{\Gamma_{(s)}^{\alpha,\beta}}{\Gamma(-\vartheta)}\mathcal{B}(\varrho+\eta s+1,\nu s-\vartheta)z^{\varrho-\vartheta}.$$

Thus, we complete the proof of this theorem. □

Theorem 9 *The following relation holds true:*

$$\mathcal{M}\left[\mathcal{D}_{z,\alpha,\beta}^{\vartheta,\wp,\eta,\nu}\left\{(1-z)^{-\kappa_1}\right\}:s\right]=\frac{\Gamma_{(s)}^{\alpha,\beta}}{\Gamma(-\vartheta)}z^{-\vartheta}\sum_{n=0}^{\infty}\frac{(\kappa_1)_n}{n!}z^n\mathcal{B}(n+\eta s+1,\nu s-\vartheta), \tag{22}$$

$$(\Re(\kappa_1>0,\ \Re(s)>0),\ \Re(\vartheta)<0,\ \alpha,\beta,\eta,\nu\in\mathbb{R}^+,\ \wp\geq 0\ and\ |z|<1).$$

Proof By using binomial expansion of $(1-z)^{-\kappa_1}$ and (21), we get

$$\begin{aligned}\mathcal{M}\left[\mathcal{D}_{z,\alpha,\beta}^{\vartheta,\wp,\eta,\nu}\left\{(1-z)^{-\kappa_1}\right\}:s\right]&=\sum_{n=0}^{\infty}\frac{(\kappa_1)_n}{n!}\mathcal{M}\left[\mathcal{D}_{z,\alpha,\beta}^{\vartheta,\wp,\eta,\nu}\left\{z^n\right\}:s\right]\\&=\frac{\Gamma_{(s)}^{\alpha,\beta}}{\Gamma(-\vartheta)}\sum_{n=0}^{\infty}\frac{(\kappa_1)_n}{n!}\mathcal{B}(n+\eta s+1,\nu s-\vartheta)z^{n-\vartheta}\\&=\frac{\Gamma_{(s)}^{\alpha,\beta}}{\Gamma(-\vartheta)}z^{-\vartheta}\sum_{n=0}^{\infty}\frac{(\kappa_1)_n}{n!}z^n\mathcal{B}(n+\eta s+1,\nu s-\vartheta).\end{aligned}$$

Hence, we achieve the desired result (22). □

7 Generating Functions for an Extended Hypergeometric Functions

In this part, we will explore using extended Appell's functions and the fractional derivative operator to construct generating functions for extended hypergeometric functions.

Theorem 10 *The following relation of extended hypergeometric function holds true:*

$$\sum_{n=0}^{\infty}\frac{(\varrho)_n}{n!}F_{\alpha,\beta}^{\wp,\eta,\nu}(\varrho+n,\kappa_1;\kappa_2;x)\tau^{\eta}=(1-\tau)^{-\varrho}F_{\alpha,\beta}^{\wp,\eta,\nu}(\varrho,\kappa_1,\kappa_2;\frac{x}{1-\tau}), \tag{23}$$

$$(\ |x|<\min\{1,|1-\tau|\}\,;\Re(\varrho)>0,\Re(\kappa_2)>\Re(\kappa_1)>0\ ).$$

Proof By using well-known identity such as

$$[(1-x)-\tau]^{-\varrho}=(1-\tau)^{-\varrho}\left(1-\frac{x}{1-\tau}\right)^{-\varrho},$$

and expand the left-hand side of above expression for $|\tau|<|1-x|$, we have

$$\sum_{n=0}^{\infty}\frac{(\varrho)_n}{n!}(1-x)^{-\varrho}\left(\frac{\tau}{1-x}\right)^n=(1-\tau)^{-\varrho}\left(1-\frac{x}{1-\tau}\right)^{-\varrho}.$$

Now, multiply both sides of above expression by x^{κ_1-1} and applying the extended fractional derivative operator $\mathcal{D}_{x,\alpha,\beta}^{\kappa_1-\kappa_2,\wp,\eta,\nu}$, we get

$$\mathcal{D}_{x,\alpha,\beta}^{\kappa_1-\kappa_2,\wp,\eta,\nu}\left\{\sum_{n=0}^{\infty}\frac{(\varrho)_n}{n!}(1-x)^{-\varrho}\left(\frac{\tau}{1-x}\right)^n x^{\kappa_1-1}\right\}$$

$$=(1-\tau)^{-\varrho}\mathcal{D}_{x,\alpha,\beta}^{\kappa_1-\kappa_2,\wp,\eta,\nu}\left\{x^{\kappa_1-1}\left(1-\frac{x}{1-\tau}\right)^{-\varrho}\right\}.$$

Interchanging the order which is valid for $\Re(\kappa_1)>0$ and $|t|<|1-x|$, we get

$$\sum_{n=0}^{\infty}\frac{(\varrho)_n}{n!}\tau^{\eta}\mathcal{D}_{x,\alpha,\beta}^{\kappa_1-\kappa_2,\wp,\eta,\nu}\left\{(1-x)^{-(\varrho+n)}x^{a-1}\right\}=(1-\tau)^{-\varrho}\mathcal{D}_{x,\alpha,\beta}^{\kappa_1-\kappa_2,\wp,\eta,\nu}\left\{\left(x^{\kappa_1-1}(1-\frac{x}{1-\tau}\right)^{-\varrho}\right\}.$$

By using (17) in the above expression we achieve the result (23). □

Theorem 11 *For the new extended hypergeometric function, the following relation holds true:*

$$\sum_{n=0}^{\infty}\frac{(\varrho)_n}{n!}F_{\alpha,\beta}^{\wp,\eta,\nu}(\rho-n,\kappa_1;\kappa_2;x)\tau^{\eta}=(1-\tau)^{-\varrho}F_{1;\alpha,\beta}^{\wp,\eta,\nu}\left(\kappa_1,\rho,\varrho;\kappa_2,x,\frac{x\tau}{1-\tau}\right). \tag{24}$$

Proof Using the following identities:

$$[1-(1-x)\tau]^{-\varrho}=(1-\tau)^{-\varrho}\left(1+\frac{x\tau}{1-\tau}\right)^{-\varrho}.$$

By using Binomial expansion to expanding the left-hand side of above relation for $|\tau|<|1-x|$, we have

$$\sum_{n=0}^{\infty}\frac{(\varrho)_n}{n!}(1-x)^n\tau^n=(1-\tau)^{-\varrho}\left(1-\frac{-x\tau}{1-\tau}\right)^{-\varrho}.$$

Now, multiplying both sides by $x^{\kappa_1-1}(1-x)^{-\rho}$ and applying the extended fractional derivative operator $\mathcal{D}_{x,\alpha,\beta}^{\kappa_1-\kappa_2,\wp,\eta,\nu}$, we have

$$\mathcal{D}_{x,\alpha,\beta}^{\kappa_1-\kappa_2,\wp,\eta,\nu}\left\{\sum_{n=0}^{\infty}\frac{(\varrho)_n}{n!}x^{\kappa_1-1}(1-x)^{-\rho+n}\tau^n\right\}$$

$$= (1-\tau)^{-\varrho} \mathcal{D}_{x,\alpha,\beta}^{\kappa_1-\kappa_2,\wp,\eta,\nu} \left\{ x^{\kappa_1-1}(1-x)^{-\rho} \left(1 - \frac{-x\tau}{1-\tau}\right)^{-\varrho} \right\}.$$

Interchanging the order which is valid for $\Re(\kappa_1) > 0$ and $|xt| < |1-x|$, we get

$$\sum_{n=0}^{\infty} \frac{(\varrho)_n}{n!} \mathcal{D}_{x,\alpha,\beta}^{\kappa_1-\kappa_2,\wp,\eta,\nu} \left\{ x^{\kappa_1-1}(1-x)^{-\rho+n}\tau^n \right\}$$

$$= (1-\tau)^{-\varrho} \mathcal{D}_{x,\alpha,\beta}^{\kappa_1-\kappa_2,\wp,\eta,\nu} \left\{ x^{\kappa_1-1}(1-x)^{-\rho} \left(1 - \frac{-x\tau}{1-\tau}\right)^{-\varrho} \right\}.$$

By using (16) and (17) in the above expression, we achieve the result (24). □

Theorem 12 *For the new extended hypergeometric function, the following relation holds true:*

$$\sum_{n=0}^{\infty} \frac{(\varrho)_n}{n!} \; F_{\alpha,\beta}^{\wp,\eta,\nu}(-n, \delta; \zeta; y) \; F_{\alpha,\beta}^{\wp,\eta,\nu}(\varrho+n, \kappa_1; \kappa_2; x) \; \tau^n$$

$$= (1-\tau)^{-\varrho} F_{2;\alpha,\beta}^{\wp,\eta,\nu} \left(\varrho, \kappa_1, \delta; \kappa_2, \zeta; \frac{x}{1-\tau}, \frac{-y\tau}{1-\tau} \right). \tag{25}$$

Proof In the expression (23), by replacing t by $(1-y)\tau$, multiplying in the resulting equality by $y^{\delta-1}$, and using extended fractional derivative operator $D_{y,\alpha,\beta}^{\delta-\zeta,p,\eta,\nu}$, we have

$$\mathcal{D}_{y,\alpha,\beta}^{\delta-\zeta,\wp,\eta,\nu} \left\{ \sum_{n=0}^{\infty} \frac{(\varrho)_n}{n!} y^{\delta-1} F_{\alpha,\beta}^{\wp,\eta,\nu}(\varrho+n, \kappa_1; \kappa_2; x)(1-y)^n \tau^n \right\}$$

$$= \mathcal{D}_{y,\alpha,\beta}^{\delta-\zeta,\wp,\eta,\nu} \left\{ (1-(1-y)\tau)^{-\varrho} \; y^{\delta-1} F_{\alpha,\beta}^{\wp,\eta,\nu} \left(\varrho, \kappa_1, \kappa_2; \frac{x}{1-(1-y)\tau} \right) \right\}.$$

Now, interchanging the order, which is valid for $|x| < 1$, $\left| \frac{1-x}{1-y} \tau \right| < 1$ and $\left| \frac{x}{1-\tau} \right| + \left| \frac{y\tau}{1-\tau} \right| < 1$, we get

$$\sum_{n=0}^{\infty} \frac{(\varrho)_n}{n!} \mathcal{D}_{y,\alpha,\beta}^{\delta-\zeta,\wp,\eta,\nu} \left\{ y^{\delta-1}(1-y)^n \right\} F_{\alpha,\beta}^{\wp,\eta,\nu}(\varrho+n, \kappa_1; \kappa_2; x) t^n$$

$$= \mathcal{D}_{y,\alpha,\beta}^{\delta-\zeta,\wp,\eta,\nu} \left\{ y^{\delta-1} \left(1 - \frac{-y\tau}{1-\tau}\right)^{-\varrho} F_{\alpha,\beta}^{\wp,\eta,\nu} \left(\varrho, \kappa_1, \kappa_2; \frac{\frac{x}{1-\tau}}{1 - \frac{-y\tau}{1-\tau}} \right) \right\}.$$

By using (16) and (19), we get the desired result. □

8 Conclusions

In this article, extended Appell's functions, Riemann–Liouville fractional derivative operators, and their Mellin transform were derived. We also studied some applications of Riemann–Liouville fractional derivative operators and with their help, we concluded some generating functions of extended hypergeometric functions. In order to justify our new operator, we know that fractional calculus is used in various engineering and science domains. With the results obtained in this article, there are several uses and applications in various fields. We come to another conclusion for further research.

For future direction, recently Ghayasuddin et al. [1] defined a new beta function using three-parameter Mittag-Leffler function and with the help of this beta functions, we derive a new generalization of Appell's functions of two- and three-variable as well as Riemann–Liouville fractional derivative operators as follows:

$$F^{\wp}_{1;\alpha,\beta,\gamma}(\kappa_1,\kappa_2,\kappa_3;\kappa_4;x,y)=\sum_{n,m=0}^{\infty}\frac{\mathcal{B}^{\wp}_{\alpha,\beta,\gamma}(\kappa_1+m+n,\kappa_4-\kappa_1)}{\mathcal{B}(\kappa_1,\kappa_4-\kappa_1)}(\kappa_2)_n(\kappa_3)_m\frac{x^n}{n!}\frac{y^m}{m!},$$

$$(\max\{|x|,|y|\}<1\ ;\ \alpha,\beta,\gamma\in\mathbb{R}^+\ \textit{and}\ \wp\geq 0).$$

$$F^{\wp}_{2;\alpha,\beta,\gamma}(\kappa_1,\kappa_2,\kappa_3;\kappa_4,\kappa_5;x,y)$$

$$=\sum_{n,m=0}^{\infty}\frac{(\kappa_1)_{m+n}\mathcal{B}^{\wp}_{\alpha,\beta,\gamma}(\kappa_2+n,\kappa_4-\kappa_2)\mathcal{B}^{\wp}_{\alpha,\beta,\gamma}(\kappa_3+m,\kappa_5-\kappa_3)}{\mathcal{B}(\kappa_2,\kappa_4-\kappa_2)\mathcal{B}(\kappa_3;\kappa_5-\kappa_3)}\frac{x^n}{n!}\frac{y^m}{m!},$$

$$(|x|+|y|<1;\quad \alpha,\beta,\gamma\in\mathbb{R}^+\ \textit{and}\ \wp\geq 0).$$

$$F^{3,\alpha,\beta,\gamma}_{\mathcal{D},\wp}(\kappa_1,\kappa_2,\kappa_3,\kappa_4;\kappa_5;x,y,z)$$

$$=\sum_{n,m,r=0}^{\infty}\frac{\mathcal{B}^{\wp}_{\alpha,\beta,\gamma}(\kappa_1+m+n+r,\kappa_5-\kappa_1)(\kappa_2)_m(\kappa_3)_n(\kappa_4)_r}{\mathcal{B}(\kappa_1,\kappa_5-\kappa_1)}\frac{x^n}{n!}\frac{y^m}{m!}\frac{z^r}{r!},$$

$$(\sqrt{|x|}+\sqrt{|y|}+\sqrt{|z|}<1\ ;\ \alpha,\beta,\gamma\in\mathbb{R}^+_0\ \textit{and}\ \wp\geq 0).$$

and

$$\mathcal{D}^{\vartheta,\wp}_{z;\alpha,\beta,\gamma}\{f(z)\}=\frac{1}{\Gamma(-\vartheta)}\int_0^z f(\tau)(z-\tau)^{-\vartheta-1}E^{\gamma}_{\alpha,\beta}\left(\frac{-\wp z^2}{\tau(z-\tau)}\right)dt$$

$$(\Re(\vartheta)<0;\ \ \wp\geq 0;\alpha,\beta,\gamma\in\mathbb{R}^+),$$

for $j-1<\Re(\vartheta)<j$ such that $j=1,2,3,....$

$$\begin{aligned}\mathcal{D}^{\vartheta,\wp}_{z;\alpha,\beta,\gamma}\{f(z)\} =&\frac{d^j}{dz^j}\mathcal{D}^{\vartheta-j}_{z}\{f(z)\}\\ =&\frac{d^j}{dz^j}\left\{\frac{1}{\Gamma(j-\vartheta)}\int\limits_0^z f(\tau)(z-1)^{-\vartheta+j-1}E^{\gamma}_{\alpha,\beta}\left(\frac{-\wp z^2}{\tau(z-\tau)}\right)d\tau\right\},\end{aligned}$$

and study their properties and applications.

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