MODEL ANALYSIS OF SYMMETRIC AND ASYMMETRIC LAMINATED CANTILEVER BEAM USING ANSYS

A Dissertation submitted
In partial fulfillment of the requirement For the Degree

MASTER OF TECHNOLOGY
In

MECHANICAL ENGINEERING

With
specialization In
Production and Industrial Engineering
By

ABU BAKAR ANSARI

(Enrollment No. 2000101580)

Under the Supervision of

Dr. MOHD ANAS
Associate Professor

DEPARTMENT OF MECHANICAL ENGINEERING
INTEGRAL UNIVERSITY, LUCKNOW

July, 2022

CERTIFICATE

It is certified that **ABU BAKAR ANSARI** (Enrollment No. 2000101580) has carried out the research work presented in this dissertation entitled **— MODEL ANALYSIS OF SYMMETRIC AND ASYMMETRIC LAMINATED CANTILEVER BEAM USING ANSYS"** for the award of **Master of Technology** from **Integral University, Lucknow** under our supervision. The dissertation embodies result of original work and studies are carried out by the student himself and the contents of the thesis do not form the basis for the award of any degree to the candidate or to any body else from this or any other university/Institution.

Dr MOHD. ANAS

Associate Professor

Department of Mechanical Engineering

Integral University, Lucknow

Date:

DECLARATION

I hereby declare that the dissertation titled –**MODEL ANALYSIS OF SYMMETRIC AND ASYMMETRIC LAMINATED CANTILEVER BEAM USING ANSYS**‖ is an authentic record of the research work carriedout by me under the supervision of Dr. MOHD. ANAS, Department of Mechanical Engineering, for the period from 2020 to 2022 at Integral University, Lucknow. No part ofthis dissertation has been presented elsewhere for any other degree or diploma earlier.

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Date -

Signature -

Name -ABU BAKAR ANSARI

Enrollment No-2000101580

ACKNOWLEDGEMENT

At this moment of accomplishment, I wish to pay my heartfelt gratitude and sincerest thanks homage to my guide, Dr. MOHD. ANAS (Associate Professor), department of mechanical engineering, Integral University, Lucknow. This work would not have been possible without his able guidance, support, and encouragement. Under his guidance, I have successfully over come many difficulties and learned a lot. He used to review my work progress, give his valuable suggestions, and made correction. His unflinching courage and conviction will always inspire me, and I hope to continue to work on his noble thought.

I am also extremely indebted to Dr. P.K.Bharti (HOD), Mechanical Engineering Department, Integral University, Lucknow for providing necessary infrastructure and resources to accomplish research work. I warmly thank to Er. Faizan Hasan and Er.Abhishek Dwivedi for their valuable advice and encouraging me at regular interval. I would also like to thank Mr. Thomas, workshop instructor for his support during my entire work.

Last but not the least, it goes without saying that I am indebted to a number of friends and well-wisher specially who have extended their co-operation and help during the work.

ABU BAKAR ANSARI

(EnrollmentNo.2000101580)

ABSTRACT

A dynamic finite element approach for free vibration analysis of generally laminated composite beams is introduced on the basis of first order shear deformation theory. The effect of Poisson effect, bending and torsional deformations, couplings among extensional, shear deformation and rotary inertia are comprised in the formulation. The dynamic stiffness matrix is defined based on the exact solutions of the differential equations of motion governing the free vibration of generally laminated composite beam. The influences of Poisson effect, material anisotropy, slender ratio, shear deformation and boundary condition on the natural frequencies of the composite beams are analyze in detail by specific carefully favored examples. The natural frequencies and mode shapes of numerical results are presented and, whenever possible, compared to those previously published solutions in order to describe the correctness and accuracy of the present approach.

Free vibration analysis of laminated composite beams is carried out using higher order shear deformation theory. Two-node, finite elements of eight degrees of freedom per node, based on the theories, are presented for the free vibration analysis of the laminated composite beams in this project work. Numerical results have been computed for various ply orientation sequence and number of layers and for various boundary conditions of the laminated composite beams and compared with the results of other higher order theories available in literature. The comparison study shows that the present considered higher order shear deformation theory forecast the natural frequencies of the laminated composite beams better than the other higher order theories considered.

For considered examples, the coding of the formulation of first order shear deformation theory and higher order shear deformation theory (two-node , finite elements of eight degree of freedom per node) done by the help of MATLAB and ANSYS 12 software package.

Keywords: - shear deformation, slender ratio, natural frequencies, higher order shear deformation theory, laminated composite beam and free vibration.

CONTENTS

Chapter 1

INTRODUCTION

INTRODUCTION

DEFINITIONS [56]:-

A composite material is defined as a material system which consists of a mixture or a combination of two or more distinctly different materials which are insoluble in each other and differ in form or chemical composition.

Thus, a composite material is labeled as any material consisting of two or more phases. Many combinations of materials termed as composite materials, such as concrete, mortar, fiber reinforced plastics, fibre reinforced metals and similar fibre impregnated materials.

Two- phase composite materials are classified into two broad categories: particulate composites and fibre reinforced composites. Particulate composites are those in which particles having various shapes and sizes are dispersed within a matrix in a random fashion. Examples as mica flakes reinforced with glass, lead particles in copper alloys and silicon carbon particles in aluminium.

Particulate composites are used for electrical applications, welding, machine parts and other purposes.

Fibre reinforced composite materials consists of fibres of significant strength and stiffness embedded in a matrix with distinct boundaries between them. Both fibres and matrix maintain their physical and chemical identities, yet their combination performs a function which cannot be done by each constituent acting singly. fibres of fibre reinforced plastics (FRP) may be short or continuous. It appears obvious that FRP having continuous fibres is indeed more efficient.

Classification of FRP composite materials into four broad categories has been done accordingly to the matrix used. They are polymer matrix composites, metal matrix composites, ceramic matrix composites and carbon/carbon composites. Polymer matrix composites are made of thermoplastic or thermoset resins reinforced with fibres such as glass, carbon or boron. A metal matrix composite consists of a matrix of metals or alloys reinforced with metal fibres such as boron or carbon. Ceramic matrix composites consist of ceramic matrices reinforced with ceramic fibres such as silicon carbide, alumina or silicon nitride. They are mainly effective for high temperature applications.

Table 1 Classification of FRP composite materials [1]

Matrix type	Fibres	Matrix
Polymer	E-glass S-glass Carbon(graphite) Kevlar Boron	Epoxy Polyimide Thermoplastics Polyester Polysulfone
Metal	Boron Carbon (graphite) Silicon carbide Alumina	Aluminium Magnesium Titanium Copper
Ceramic	Silicon carbide Alumina Silicon nitride	Silicon carbide Alumina Glass ceramic Silicon nitride
Carbon	Carbon	Carbon

Of all the types of composites discussed above, the most important is the fibre reinforced composites this is form the application point of view. This project is deal with fibre reinforced polymer matrix composite materials.

Fibres[56] :-

Materials in fibre form are stronger and stiffer than that used in a bulk form. There is a likely presence of flaws in bulk material which affects its strength while internal flaws are mostly absent in the case of fibres. Further , fibres have strong molecular or crystallographic alignment and are in the shape of very small crystals. Fibres have also a low density which is disadvantageous.

Fibres is the most important constituent of a fibre reinforced composite material. They also occupy the largest volume fraction of the composite. Reinforcing fibres as such can take up only its tensile load. But when they are used in fibre reinforced composites, the surrounding matrix enables the fibre to contribute to the major part of the tensile, compressive, and flexural or shear strength and stiffness of FRP composites.

Glass fibres[56]

The most common fibre used in polymeric fibre reinforced composites is the glass fibre. The main advantage of the glass fibre is its low cost. Its other advantage are its high tensile strength, low chemical resistance and excellent insulating properties. Among its disadvantages are its low tensile modulus somewhat high specific gravity, high degree of hardness and reduction of tensile strength due to abrasion during handling. Moisture decreases the glass fibre strength. Glass fibres are susceptible to sustained loads, as they cannot withstand loads for long periods.

Two types of glass fibres are used in FRP industries. They are E-glass and S-glass . E-glass has the lowest cost among all fibres.

S-glass has high tensile strength. Its typical composition is 65% SiO_2 , 25% Al_2O_3 and 10% MgO. The cost of s-Glass is 20-30 times that of E-glass. The tensile strength of S-glass is 33% greater and the modulus of elasticity is 20% higher than that of E-glass. The principal advantages of S-glass are its high strength-to-weight ratio, its superior strength relation at elevated temperature and its high fatigue limit. In spite of its high cost, its main application area is in aerospace components such as rocket mortars.

Carbon fibres[56]

Carbon fibres are charactyerised by a combination of high strength, high stiffness and light weight. The advantages of carbon fibres are their very high tensile strength-to-weight ratio, high tensile modulus-to-weight ratio, very low coefficient of thermal expansion and high fatigue strength. The disadvantages are their low impact resistance and high electrical conductivity. Due to the high cost the use of the carbon fibres is justified only in weight critical structures, that is mostly applied to aerospace industry.

Aramid fibres[56]

Kevlar aramid is made of carbon , hydrogen, oxygen and nitrogen and is essentially an aromatic organic compound. The advantages of aramid fibres are their low density, high tensile strength and low cost.

Characteristics of Kevlar 49 are its high strength and stiffness, light weight, vibration damping, resistance to damage, fatigue and stress ruptures. Another variety Kevlar 29 which is of low density and high strength. Kevlar 29 is used in ropes, cables and coated fabrics for inflatables.

The principal disadvantages of aramid fibres are their low compressive strength and the difficulty in cutting or machining. For structures or structural components where compression and bending are predominant such as in a shell, aramid fibres can be used only when it is hybridized with glass or carbon fibres.

A more advanced variety of Kevlar fibre is Kevlar 149. Of all commercially available aramid fibres, it has the highest tensile modulus as it has 40% higher modulus than Kevlar 49. The strain at failure for Kevlar 149 is; however, lower than that of Kevlar 49. Aramid fibres are costlier than E-glass, but are cheaper than carbon fibres.

Boron fibres[56]

Boron fibres are characterized by their very high tensile modulus. Boron fibres have relatively large diameters and due to this they are capable of withstanding large compressive stress and providing excellent resistance to buckeling. Boron fibres are , however , costly and in fact are costlier than most varities of carbon fibres. The application area of boron fibres at present is restricted to aerospace industries only.

Ceramic fibres[56]

Ceramic fibres are mainly used in application areas dealing with elevated temperature. Examples of ceramic fibres are silicon carbide and aluminium oxide. Ceramic fibres has an advantage in that they have properties such as high strength, high elastic modulus with high temperature capabilities and are free from environmental attack.

Polymeric matrix[56]

Polymers are divided into two broad categories: thermoplastic and thermoset. Thermoplastic polymers are those which are heat softened ,melted and reshaped as many times as desired. But a thermoset polymer cannot be melted or reshaped by the application of heat or pressure.

The advantages of thermoplastic matrices are their improved fracture toughness over the thermoset matrix and their potential of much lower cost in the manufacturing of finished composites.

Traditionally, thermoset polymers are widely used as a matrix material for fibre reinforced composites in structural composite components. Thermoset polymers improve thermal stability and chemical resistance.

For the purpose of a simple classification, we may divide the thermosets into five categories:-

(1) Polyster resin,
(2) Epoxy resin,
(3) Vinyl ester resin,
(4) Phenolic resin and
(5) High performance resin.

Polyster resins[56]

The most commonly used resin in glass reinforced plastic construction is the polyster resin and they have exhibited good performance. The main advantages of polyester resins are their reasonable cost and ease with which they can be used.

Epoxy resins[56]

Epoxy resins are mostly used in aerospace structures for high performance applications. It is also used in marine structures, rarely though, as cheaper varieties of resins other than epoxy are available**.** the extensive use of epoxy resins in industry is due to :- (1) the ease with which it can be processed,(2) excellent mechanical properties and (3) high hot and wet strength properties .

Vinyl ester resins[56]

Vinyl ester resin is superior to polyester resin because it offers greater resistant to water. These resins provide superior chemical résistance and superior retention properties of strength and stiffness at elevated temperature. In construction and marine industries , vinyl ester resins have been widely used in boat construction.

Phenolic resins[56]

The main characteristics of phenolic resins are their excellent fire resistance properties. As such they are now introduced in high temperature application areas. The recently developed cold-cure varieties of phenolic resins are used for contact moulding of structural laminates.

Phenolic resins have inferior mechanical properties to both polyester resins and epoxy resins, but have higher maximum operating temperature, much better flame retardant and smoke and toxic gas emission characteristics. Due to the above advantages , phenolic resins are the only matrix used in aircraft interior.phenolic resins are increasingly used in internal bulkheads, decks and furnishings in ships.

Application of composites

1. Marine field
2. Aircraft and Space
3. Automotive
4. Sporting goods
5. Medical Devices
6. Commercial applications.

The Timoshenko Beam Theory

It is well known the classical theory of Euler–Bernoulli beam assumes that-

(1) the cross-sectional plane perpendicular to the axis of the beam remains plane after deformation (assumption of a rigid cross-sectional plane);

(2) The deformed cross-sectional plane is still perpendicular to the axis after deformation.

The classical theory of beam neglects transverse shearing deformation where the transverse shear stress is determined by the equations of equilibrium. It is applicable to a thin beam. For a beam with short effective length or composite beams, plates and shells, it is inapplicable to neglect the transverse shear deformation. In 1921, Timoshenko presented a revised beam theory considering shear deformation1 which retains the first assumption and satisfies the stress-strain relation of shear.

Advantages:-

1. High resistance to fatigue and corrosion degradation.
2. High ‗strength or stiffness to weight' ratio.
3. High resistance to impact damage.
4. Improved friction and wear properties.
5. Improved dent resistance is normally achieved. Composite panels do not sustain damage as easily as thin gage sheet metals.
6. Due to greater reliability, there are fewer inspections and structural repairs.

PRESENT INVESTIGATION

In the current investigation the main objective is to find out the free vibration of generally laminated composite beams based on first-order shear deformation theory and derived through the use of Hamilton‘s principle. The Poisson effect, rotary inertia, shear deformation and material coupling among the bending, extensional and torsional deformations are embraced in the formulation. A dynamic stiffness matrix is made to solve the free vibration of the generally laminated composite beams [54]. The dynamic finite element method deals with the mass distribution within a beam element exactly and thus it provides accurate dynamic characteristics of a composite beam. Natural frequencies and mode shapes are obtained for the generally laminated composite beams. The natural frequencies are investigated and comparisons of the current results with the available solutions in literature are presented [54].

Also the current investigation is based on the higher order shear deformation theories, for the dynamic analysis of the simply supported laminated composite beam. Numerical results have been computed for various boundary conditions for the homogeneous and laminated composites beams and the numerical results are compared with the results of other theories available in literature [54, 55].

After the comparisons of results we noticed that the theories predict the natural frequencies of the beams better than the other higher order shear deformation theories [55].

Apart from the presentation of analytical solutions to the vibration problems of the composite laminated beams, two node finite elements of eight degrees of freedom per node are also investigated in this present analysis to determine the natural frequencies of simply-supported and clamped- free laminated composite beams for which analytical solutions cannot be obtained using the higher order shear deformation theories [55]. Numerical results obtained for the above problems compared to the analytical and finite element solutions available in the literature [54, 55].

The present results are compared with solutions available in the literature and obtained by the help of MATLAB and ANSYS software.

Chapter 2

LITERATURE REVIEW

LITERATURE REVIEW

The fiber-reinforced composite materials are ideal for structural applications where high strength-to-weight and stiffness-to-weight ratios are required. Composite materials can be tailored to meet the particular requirements of stiffness and strength by altering lay-up and fiber orientations. The ability to tailor a composite material to its job is one of the most significant advantages of a composite material over an ordinary material. So the research and development of composite materials in the design of aerospace, mechanical and civil structures has grown tremendously in the past few decades. It is essential to know the vibration characteristics of these structures, which may be subjected to dynamic loads in complex environmental conditions. If the frequency of the loads variation matches one of the resonance frequencies of the structure, large translation/torsion deflections and internal stresses can occur, which may lead to failure of structure components. A variety of structural components made of composite materials such as aircraft wing, helicopter blade, vehicle axles, and turbine blades can be approximated as laminated composite beams, which requires a deeper understanding of the vibration characteristics of the composite beams. The practical importance and potential benefits of the composite beams have inspired continuing research interest. A number of researchers have been developed numerous solution methods in recent 20 years.

Raciti and Kapania [2] collected a report of developments in the vibration analysis of laminated composite beams.

Chandrashekhara et al. [3] found the accurate solutions based on first order shear deformation theory including rotary inertia for symmetrically laminated beams.
The laminated beams by a systematic reduction of the constitutive relations of the three-dimensional anisotropic body and found the basic equations of the beam theory based on the parabolic shear deformation theory represented by Bhimaraddi and Chandrashekhara [4].

A third-order shear deformation theory for static and dynamic analysis of an orthotropic beam incorporating the impact of transverse shear and transverse normal deformations developed by Soldatos and Elishakoff [5].

The exact solutions for symmetrically laminated composite beams with 10 different boundary conditions, where shear deformation and rotary inertia were considered in the analysis developed by Abramovich [6].

Hamilton's principle to calculate the dynamic equations governing the free vibration of laminated composite beams. The impacts of transverse shear deformation and rotary inertia were included, and analytical solutions for unsymmetrical laminated beams were obtained by applying the Lagrange multipliers method developed by Krishnaswamy et al. [7].

The free vibration behavior of laminated composite beams by the conventional finite element analysis using a higher-order shears deformation theory. The Poisson effect, coupled extensional and bending deformations and rotary inertia are considered in the formulation studied by Chandrashekhara and Bangera [8].

Abramovich and Livshits [9] presented the free vibration analysis of non-symmetric cross-ply laminated beams based on first-order shear deformation theory.

Khdeir and Reddy [10] evolved the analytical solutions of various beam theories to study the free vibration behavior of cross-ply rectangular beams with arbitrary boundary conditions.

Biaxial bending, axial and torsional vibrations using the finite element method and the first-order shear deformation theory examined by Nabi and Ganesan [11].

The analytical solutions for laminated beams based on first-order shear deformation theory including rotary inertia obtained by Eisenberger et al. [12].

Banerjee and Williams [13] evolved the exact dynamic stiffness matrix for a uniform, straight, bending–torsion coupled, composite beam without the effects of shear deformation and rotary inertia included.

Teboub and Hajela [14] approved the symbolic computation technique to analyze the free vibration of generally layered composite beam on the basis of a first-order shear deformation theory. The model used considering the effect of Poisson effect, coupled extensional, bending and torsional deformations as well as rotary inertia.

An exact dynamic stiffness matrix for a composite beam with the impacts of shear deformation, rotary inertia and coupling between the bending and torsional deformations included presented by Banerjee and Williams [15].

An analytical method for the dynamic analysis of laminated beams using higher order refined theory developed by Kant et al. [16] .

Shimpi and Ainapure [17] presented the free vibration of two-layered laminated cross-ply beams using the variation ally consistent layer wise trigonometric shear deformation theory.

The in-plane and out-of-plane free vibration problem of symmetric cross-ply laminated composite beams using the transfer matrix method analyzed by Yildirim et al. [18].

Yildirim et al. [19] examined the impacts of rotary inertia, axial and shear deformations on the in-plane free vibration of symmetric cross-ply laminated beams.

The stiffness method for the solution of the purely in-plane free vibration problem of symmetric cross-ply laminated beams with the rotary inertia, axial and transverse shear deformation effects included by the first-order shear deformation theory developed by Yildirim [20].

Mahapatra et al. [21] presented a spectral element for Bernoulli–Euler composite beams.

Ghugal and Shimpi [22] preposed a review of displacement and stress-based refined theories for isotropic and anisotropic laminated beams and discussed various equivalent single layer and layer wise theories for laminated beams.

Higher-order mixed theory for determining the natural frequencies of a diversity of laminated Simply-Supported beams presented by Rao et al. [23] .

A new refined locking free first-order shear deformable finite element and demonstrated its utility in solving free vibration and wave propagation problems in laminated composite beam structures with symmetric as well as asymmetric ply stacking proposed by Chakraborty et al. [24].

A spectral finite element model for analysis of axial– flexural–shear coupled wave propagation in thick laminated composite beams and derived an exact dynamic stiffness matrix proposed by Mahapatra and Gopalakrishnan [25].

A new approach combining the state space method and the differential quadrature method for freely vibrating laminated beams based on two-dimensional theory of elasticity proposed by Chen et al. [26].

Chen et al. [27] reported a new method of state space-based differential quadrature for free vibration of generally laminated beams.

Ruotolo [28] proposed a spectral element for anisotropic, laminated composite beams. The axial-bending coupled equations of motion were derived under the assumptions of the first-order shear deformation theory and the spectral element matrix was formulated.

A two-noded curved composite beam element with three degrees-of-freedom per node for the analysis of laminated beam structures. The flexural and extensional deformations together with transverse shear deformation based on first-order shear Deformation theories were incorporated in the formulation. Also, the Poisson effect was incorporated in the formulation in the beam constitution equation presented by Raveendranath et al. [29].

A complete set of equations governing the dynamic behavior of pre-twisted composite space rods under isothermal conditions based on the Timoshenko beam theory. The anisotropy of the rod material, the curvatures of the rod axis, and the effects of the rotary inertia, the shear, axial deformations and Poisson effect were considered in the formulation reported by Yildirim [30] .

Banerjee [31,32] reported the exact expressions for the frequency equation and mode shapes of composite Timoshenko beams with cantilever end conditions. The impacts of material coupling between the bending and torsional modes of deformation together with the effects of shear deformation and rotary inertia was taken into account when formulating the theory.

Bassiouni et al. [33] proposed a finite element model to investigate the natural frequencies and mode shapes of the laminated composite beams. The model needed all lamina had the same lateral displacement at a typical cross-section, but allowed each lamina to rotate a different amount from the other. The transverse shear deformation was included.

A new variational consistent finite element formulation for the free vibration analysis of composite beams based on the third-order beam theory proposed by Shi and Lam [34].

Chen et al. [35] presented a state space method combined with the differential quadrature method to examined the free vibration of straight beams with rectangular cross-sections on the basis of the two-dimensional elasticity equations with orthotropy.

The vibration analysis of cross-ply laminated beams with different sets of boundary conditions based on a three degree-of-freedom shear deformable beam theory. The Ritz method was adopted to determine the free vibration frequencies presented by Aydogdu [36].

A refined two-node, 4 DOF/node beam element based on higher-order shear deformation theory for axial–flexural– shear coupled deformation in asymmetrically stacked laminated composite beams. The shape function matrix used by the element satisfied the static governing equations of motion developed by Murthy et al. [37].

The free vibration behavior of symmetrically laminated fiber reinforced composite beams with different boundary conditions. The impacts of shear deformation and rotary inertia were considered and the finite-difference method was used to solve the partial differential equations describing the free vibration motion analyzed by Numayr et al. [38].

The free vibration analysis of laminated composite beams using two higher-order shear deformation theories and finite elements based on the theories. Both theories considered a quintic and quartic variation of in plane and transverse displacements in the thickness coordinates of the beams, respectively, and satisfied the zero transverse shear strain/stress conditions at the top and bottom surfaces of the beams developed by Subramanian [39] .

A new layer wise beam theory for generally laminated composite beam and contrasted the analytical solutions for static bending and free vibration with the three-dimensional elasticity solution of cross-ply laminates in cylindrical bending and with three-dimensional finite element analysis for angle-ply laminates developed by Tahani [40] .

A 21 degree-of-freedom beam element, based on the FSDT, to study the static response, free vibration and buckling of unsymmetrical laminated composite beams. They enlisted an accurate model to obtain the transverse shear correction factor preposed by Goyal and Kapania [41].

Finite elements have also been developed based on Timoshenko beam theory [42]. Most of the finite element models developed for Timoshenko beams possess a two node-two degree of freedom structure based on the requirements of the variation principle for the Timoshenko‘s displacement field.

A Timoshenko beam element showing that the element converged to the exact solution of the elasticity equations for a simply supported beam provided that the correct value of the shear factor was used proposed by Davis et al. [43].

Thomas et al. [44] proposed a new element of two nodes having three degrees of freedom per node, the nodal variables being transverse displacement, shear deformation and rotation of cross-section. The rates of convergence of a number of the elements were compared by calculating the natural frequencies of two cantilever beams. Further this paper gave a brief summary of different Timoshenko beam elements.

For the first time a finite element model with nodal degrees of freedom which could satisfy all the forced and natural boundary conditions of Timoshenko beam. The element has degrees of freedom as transverse deflection, total slope (slope due to bending and shear deformation), bending slope and the first derivative of the bending slope presented by Thomas and Abbas [45].

A second-order beam theory requiring two coefficients, one for cross-sectional warping and the other for transverse direct stress, was developed by Stephen and Levinson [46].

A beam theory for the analysis of the beams with narrow rectangular cross-section and showed that his theory predicted better results when compared with elasticity solution than Timoshenko beam theory. Though this required no shear correction factor, the approach followed by him to derive the governing differential equations was variationally inconsistent developed by Levinson [47].

Later Bickford [48] represented Levinson theory using a variational principle and also showed how one could obtain the correct and variationally consistent equations using the vectorial approach. Thus the resulting differential equation for consistent beam theory is of the sixth order, whereas that for the inconsistent beams theory is of the fourth-order.

An improved theory in which the in-plane displacement was assumed to be cubic variation in the thickness coordinate of the beam whereas the transverse displacement was assumed to be the sum of two partial deflections, deflection due to bending and deflection due to transverse shear. This theory does not impacts the effect of transverse normal strain and does not satisfy the zero strain/stress conditions at the top and bottom surfaces of the beam reported by Krishna Murty [49] .

A higher order beam finite element for bending and vibration problems of the beams. In this formulation, the theory imagines a cubic variation of the in-plane displacement in thickness co-ordinate and a parabolic variation of the transverse shear stress across the thickness of the beam. Further the theory satisfies the zero shear strain conditions at the top and bottom surfaces of the beam and neglects the effect of the transverse normal strain developed by Heyliger and Reddy [50].

A C^0 finite element model based on higher order shear deformation theories including the effect of the transverse shear and normal strain and the finite element fails to satisfy the zero shear strain conditions at the top and bottom surfaces of the beam proposed by Kant and Gupta [51].

The free vibration analysis of the laminated composite beams using a set of three higher order shear deformation theories and their corresponding finite elements. These theories also fail to satisfy the zero-strain conditions at the top and bottom surfaces of the beams. Further the impacts of the transverse normal strain were not included in the theories investigated by Marrur and Kant [52].

An analytical solution to the dynamic analysis of the laminated composite beams using a higher order refined theory. This model also fails to satisfy the traction- free surface conditions at the top and bottom surfaces of the beam but has included the effect of transverse normal strain preposed by Kant et al. [53].

2.1 OUTLINE OF THE PRESENT WORK:-

In the current investigation the main objective is to find out the free vibration of generally laminated composite beams based on first-order shear deformation theory and derived through the use of Hamilton‘s principle. The Poisson effect, rotary inertia, shear deformation and material coupling among the bending, extensional and torsional deformations are embraced in the formulation. A dynamic stiffness matrix is made to solve the free vibration of the generally laminated composite beams. The dynamic finite element method deals with the mass distribution within a beam element exactly and thus it provides accurate dynamic characteristics of a composite beam. Natural frequencies and mode shapes are obtained for the generally laminated composite beams. The natural frequencies are investigated and comparisons of the current results with the available solutions in literature are presented. For the results software package using for the coding and programming is MATLAB and ANSYS 12.

This thesis contains seven chapters including this chapter.
A detailed survey of relevant literature is reported in chapter 2.

In chapter 3 dynamic analysis of laminated composite beam using first order shear deformation theory and higher order theories including finite element method formulation is carried out common boundary conditions, such as clamped-free, simply supported, clamped-clamped and Clamped- simply supported has been analyzed.

In chapter 4 details of computational approach have been outlined. How to coding in MATLAB software and ANSYS 12 have been outlined step – by- step procedure.

In chapter 5 important results (natural frequencies and mode shapes) drawn from the present investigations reported in chapters 3 and 4, this chapter including results obtained by MATLAB and ANSYS 12 for the various boundary condition of laminated composite beam.

Finally in chapter 6 important conclusions drawn from the present investigations reported in chapters 3-5 along with suggestions for further work have been presented.

Chapter 3

THEORY AND FORMULATION

FIRST ORDER SHEAR DEFORMATION THEORY [54]

MATHEMATICAL FORMULATION

A generally laminated composite beam, is made of many piles of orthotropic materials, principal material axis of a ply may be oriented at an angle with respect to the x axis. Consider the origin of the beam is on mid-plane of the beam and x-axis coincident with the beam axis. As shown in fig.1,

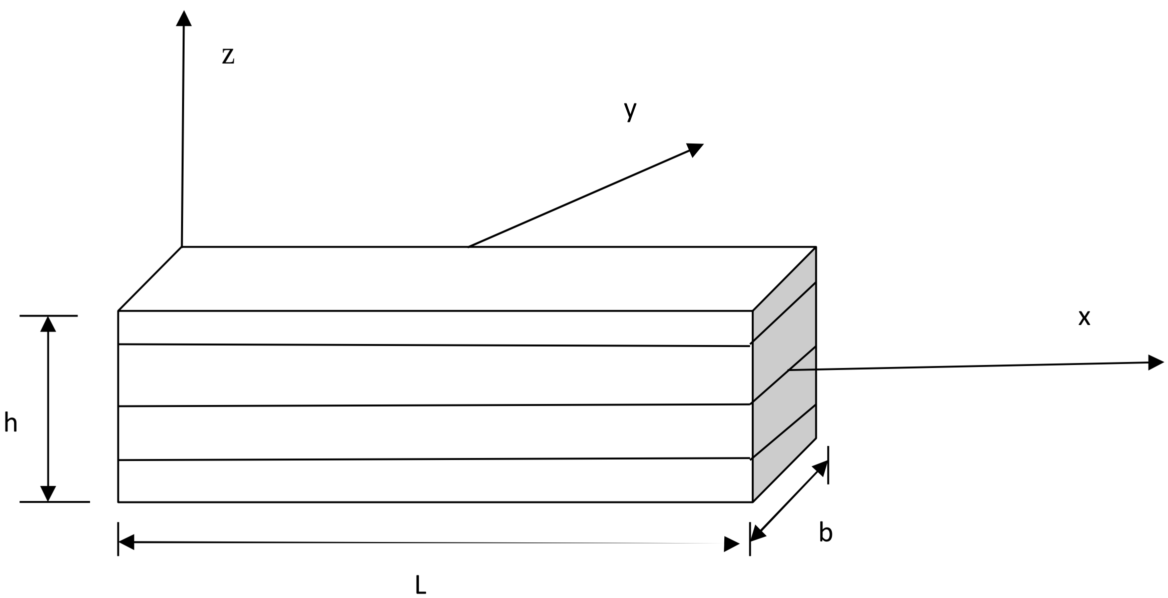

Fig. 1. Geometry of a laminated composite beam [54]

Where,

L= length of the beam,

b= breadth of the beam,

h= thickness of the beam.

Based on first- order shear deformation theory, assumed displacement field for the laminated composite beam can be written as-

$$u(x, z, t) = u_0(x, t) + z\theta(x, t), \qquad (1\ i)$$

$$v(x, z, t) = z\psi(x, t), \qquad (1j)$$

$$w(x,z,t) = w_0(x,t), \tag{1k}$$

where, u_0=axial displacements of a point on the mid plane in the x-directions,

w_0= axial displacements of a point on the mid plane in the z-directions

θ = rotation of the normal to the mid-plane about the y axis,

ψ = rotation of the normal to the mid-plane about the y axis,

t = time.

The strain-displacement relations are given by- (by theory of elasticity)—

$$\varepsilon_x = \partial u_0 / \partial x + z\partial\theta / \partial x \tag{2i}$$

$$\gamma_{xz} = \partial w_0 / \partial x \tag{2j}$$

$$\gamma_{xy} = \partial\psi / \partial x \tag{2k}$$

$$k_x = \partial\theta / \partial x \tag{2l}$$

$$k_{xy} = \partial\psi / \partial x \tag{2m}$$

By the classical lamination theory, the constitutive equations of the laminate can be obtained as-

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{3}$$

Where,(I,j=1,2,6)

N_x, N_y and N_{xy} are the in-plane forces,

M_x, M_y and M_{xy} are the bending and twisting moments,

ε_x, ε_y and γ_{xy} are the mid-plane strains,

κ_x, κ_y and κ_{xy} are the bending and twisting curvatures,

A_{ij}, B_{ij} and D_{ij} are the extensional stiffnesses, coupling stiffnesses and bending stiffnesses, respectively.

for the case of laminated composite beam,

N_y and N_{xy}, the in-plane forces and the bending moment $M_y = 0$.

ε_y^0, γ_{xy} and the curvature κ_y assumed to be non-zero

Then, now equation (3) can be rewritten as-

$$\begin{Bmatrix} N_x \\ M_x \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & B_{11} & B_{16} \\ B_{11} & D_{11} & D_{16} \\ B_{16} & D_{16} & D_{66} \end{bmatrix} \begin{Bmatrix} \partial u_0 / \partial x \\ \partial \theta / \partial x \\ \partial \psi / \partial x \end{Bmatrix} \tag{4}$$

Now considering the effect of transverse shear deformation then,-

$$Q_{xz} = A_{55}\gamma_{xz} = A_{55}\left(\partial w_0 / \partial x + \theta\right), \tag{5}$$

Where,-

Q_{xz} is the transverse shear force per unit length and

$$\begin{bmatrix} \overline{A}_{11} & \overline{B}_{11} & \overline{B}_{16} \\ \overline{B}_{11} & \overline{D}_{11} & \overline{D}_{16} \\ \overline{B}_{16} & \overline{D}_{16} & \overline{D}_{66} \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} & B_{16} \\ B_{11} & D_{11} & D_{16} \\ B_{16} & D_{16} & D_{66} \end{bmatrix} - \begin{bmatrix} A_{12} & A_{16} & B_{12} \\ B_{12} & B_{16} & D_{12} \\ B_{26} & B_{66} & D_{26} \end{bmatrix} * \begin{bmatrix} A_{22} & A_{26} & B_{22} \\ A_{26} & A_{66} & B_{26} \\ B_{22} & B_{26} & D_{22} \end{bmatrix}^{-1} * \begin{bmatrix} A_{12} & A_{16} & B_{12} \\ B_{12} & B_{16} & D_{12} \\ B_{26} & B_{66} & D_{26} \end{bmatrix}^{T} \quad (6)$$

The laminate stiffness coefficients Aij , Bij , Dij (I,j= 1,2,6) and the transverse shear stiffness

A_{55} which are functions of laminate ply orientation , material properties and stack sequences,

are given as-

$$A_{ij} = \int_{-h/2}^{h/2} \overline{Q_{ij}} dz$$

$$B_{ij} = \int_{-h/2}^{h/2} \overline{Q_{ij}} \, z dz$$

$$D_{ij} = \int_{-h/2}^{h/2} \overline{Q_{ij}} \, z^2 dz \quad (7)$$

$$A_{55} = k \int_{-h/2}^{h/2} \overline{Q_{55}} \, dz \quad (8)$$

Where,-

k is the shear correction factor.

The transformed reduced stiffness constants Qij (i,j=1,2,6) are given as-

$$\overline{Q_{11}} = \left(Q_{11}\cos^4\varphi + 2\left(Q_{12} + 2Q_{66}\right)\sin^2\varphi\cos^2\varphi + Q_{22}\cos^2\varphi\right) \qquad (9i)$$

$$\overline{Q_{12}} = \left(Q_{11} + Q_{22} - 4Q_{66}\right)\sin^2\varphi\cos^2\varphi + Q_{12}\left(\sin^4\varphi + \cos^4\varphi\right) \qquad (9j)$$

$$\overline{Q_{22}} = \left(Q_{11}\sin^4\varphi + 2\left(Q_{12} + 2Q_{66}\right)\sin^2\varphi\cos^2\varphi + Q_{22}\cos^4\varphi\right) \qquad (9k)$$

$$\overline{Q_{16}} = \left(Q_{11} - Q_{22} - 2Q_{66}\right)\sin\varphi\cos^3\varphi + \left(Q_{12} - Q_{22} + 2Q_{66}\right)\cos\varphi\sin^3\varphi \qquad (9l)$$

$$\overline{Q_{26}} = \left(Q_{11} - Q_{22} - 2Q_{66}\right)\sin^3\varphi\cos\varphi + \left(Q_{12} - Q_{22} + 2Q_{66}\right)\cos^3\varphi\sin\varphi \qquad (9m)$$

$$\overline{Q_{66}} = \left(Q_{11} - Q_{22} - 2Q_{66} - 2Q_{12}\right)\sin^2\varphi\cos^2\varphi + Q_{66}\left(\sin^4\varphi + \cos^4\varphi\right) \qquad (9n)$$

$$\overline{Q_{55}} = G_{13}\cos^2\varphi + G_{23}\sin^2\varphi \qquad (9o)$$

Where,-

∅ is the angle between the fiber direction and longitudinal axis of the beam.

The reduced stiffness constants Q_{11} , Q_{12} , Q_{22} and Q_{66} can be obtained in terms of the engineering constants[57]-

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}},$$

$$Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}},$$

$$Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}},$$

$$Q_{66} = G_{12} \qquad (10)$$

The total strain energy V of the laminated composite beam given as—

$$V = \frac{1}{2}\int_0^L \left[N_x \varepsilon_x^0 + M_x \kappa_x + M_{xy} \kappa_{xy} + Q_{xz} \gamma_{xz} \right] b.dx \qquad (11)$$

Substituting $\varepsilon_x^0, \kappa_x, \kappa_{xy}$ and γ_{xz} values from equation (2) into equation (11) then –

$$V = \frac{1}{2}\int_0^L \left[N_x \partial u_0 / \partial x + M_x \partial\theta / \partial x + M_{xy} \partial\psi / \partial x + Q_{xz} \left(\partial w_0 / \partial x + \theta \right) \right] b.dx \qquad (12)$$

Total kinetic energy T of the laminated composite beam is given as –

$$T = \frac{1}{2}\int_0^L \int_{-h/2}^{h/2} \rho \left[\left(\partial u / \partial t\right)^2 + \left(\partial v / \partial t\right)^2 + \left(\partial w / \partial t\right)^2 \right] b.dx.dz \qquad (13)$$

Where,-

ρ is the mass density per unit volume.

Now substituting u, ν and ω from equation (1) into equation (13) and after integration with respect to z we get—

$$T = \frac{1}{2}\int_0^L \left[I_1 \left(\partial u_0 / \partial t\right)^2 + I_3 \left(\partial\theta / \partial t\right)^2 + 2I_2 \left(\partial u_0 / \partial t\right)\left(\partial\theta / \partial t\right) + I_3 \left(\partial\psi / \partial t\right)^2 + I_1 \left(\partial w_0 / \partial t\right)^2 \right] b.dx \qquad (14)$$

Where,-

$$I_1 = \int_{-h/2}^{h/2} \rho dz$$

$$I_2 = \int_{-h/2}^{h/2} \rho \, z dz$$

$$I_3 = \int_{-h/2}^{h/2} \rho \, z^2 dz \qquad (15)$$

By the use of Hamilton's principle, the governing equations of motion of the laminated composite beam can be expressed in the form-

$$\int_{t_1}^{t_2} \left(\delta T - \delta V\right) dt = 0 \tag{16}$$

At $t = t_1$ and t_2 -

$$\delta u_0 = \delta w_0 = \delta\theta = \delta\psi = 0$$

After substitution the variational operations yields the following governing equation of motion-

$$-I_1\left(\frac{\partial^2 u_0}{\partial t^2}\right) - I_2\left(\frac{\partial^2 \theta}{\partial t^2}\right) + A_{11}\left(\frac{\partial^2 u_0}{\partial x^2}\right) + B_{11}\left(\frac{\partial^2 \theta}{\partial x^2}\right) + B_{16}\left(\frac{\partial^2 \psi}{\partial x^2}\right) = 0 \tag{17i}$$

$$-I_1\left(\frac{\partial^2 w_0}{\partial t^2}\right) + A_{55}\left(\frac{\partial^2 w_0}{\partial x^2}\right) + A_{55}\left(\frac{\partial \theta}{\partial x}\right) = 0 \tag{17j}$$

$$-I_3\left(\frac{\partial^2 \theta}{\partial t^2}\right) - I_2\left(\frac{\partial^2 u_0}{\partial t^2}\right) + \overline{B}_{11}\left(\frac{\partial^2 u_0}{\partial x^2}\right) + \overline{D}_{11}\left(\frac{\partial^2 \theta}{\partial x^2}\right) + \overline{D}_{16}\left(\frac{\partial^2 \psi}{\partial x^2}\right) - A_{55}\left(\frac{\partial w_0}{\partial x}\right) - A_{55}\theta = 0 \tag{17k}$$

$$-I_3\left(\frac{\partial^2 \psi}{\partial t^2}\right) + B_{16}\left(\frac{\partial^2 u_0}{\partial x^2}\right) + D_{16}\left(\frac{\partial^2 \theta}{\partial x^2}\right) + D_{66}\left(\frac{\partial^2 \psi}{\partial x^2}\right) = 0 \tag{17l}$$

Note:-

If the Poisson effect is ignored, the coefficients in equation (17)[$\overline{A}_{11}$, $\overline{B}_{11}$, $\overline{B}_{16}$, $\overline{D}_{11}$, $\overline{D}_{16}$, $\overline{D}_{66}$] should be replaced by the laminate stiffness coefficients [$A_{11}, B_{11}, B_{16}, D_{11}, D_{16}, D_{66}$].

Equation (17) have solutions that are separable in time and space, and that the time dependence is harmonic, like as-

$$u_0(x,t) = U(x)\sin \omega t, \tag{18i}$$

$$w_0(x,t) = W(x)\sin \omega t, \tag{18j}$$

$$\theta(x,t) = \Theta(x)\sin \omega t, \tag{18k}$$

$$\psi(x,t) = \Psi(x)\sin \omega t, \tag{18l}$$

Where,

ω is the angular frequency,

$U(x)$, $W(x)$, $\Theta(x)$ and $\Psi(x)$ are the amplitudes of the sinusoidally varying longitudinal displacement, bending displacement, normal rotation and torsional rotation respectively.

Now after the substitution of equation (18) values in equation (17), the following differential eigenvalue problem is obtained:-

$$\omega^2 I_1 U + \omega^2 I_2 \Theta + \bar{A}_{11} U' + \bar{B}_{11} \Theta' + \bar{B}_{16} \Psi' = 0 \tag{19i}$$

$$\omega^2 I_1 W + A_{55} W' + A_{55} \Theta' = 0 \tag{19j}$$

$$\omega^2 I_3 \Theta + \omega^2 I_2 U + \bar{B}_{11} U' + \bar{D}_{11} \Theta' + \bar{D}_{16} \Psi' - A_{55} W' - A_{55} \Theta = 0 \tag{19k}$$

$$\omega^2 I_3 \Psi + \bar{B}_{16} U' + \bar{D}_{16} \Theta' + \bar{D}_{66} \Psi' = 0 \tag{19l}$$

Where the superscript primes denote the derivatives with respect to x.

The solution to equation (19) are given by-

$$U(x) = \bar{A} e^{\kappa x}$$

$$W(x) = \overline{B}e^{\kappa x}$$

$$\Theta(x) = \overline{C}e^{\kappa x}$$

$$\Psi(x) = \overline{D}e^{\kappa x} \tag{20}$$

After the substitution of equation (20) into equation (19) the equivalent algebraic eigenvalue equations are obtained and the equations have non-trivial solutions when the determinant of the coefficient matrix of $\overline{A}, \overline{B}, \overline{C}$ and $\overline{D}$ vanishes. Now consider that determinant is zero , then the characteristics equations, which is an eight-order polynomial equation in κ :-

$$\eta_4 \kappa^8 + \eta_3 \kappa^6 + \eta_2 \kappa^4 + \eta_1 \kappa^2 + \eta_0 = 0 \tag{21}$$

Where:-,

$$\eta_4 = -A_{55}\left(\overline{B}_{16}\overline{D}_{11} - 2\overline{B}_{11}\overline{B}_{16}\overline{D}_{16} + \overline{B}_{11}^2\overline{D}_{66} + \overline{A}_{11}\left(\overline{D}_{16}^2 - \overline{D}_{11}\overline{D}_{66}\right)\right) \tag{22i}$$

$$\eta_3 = -\left(\begin{array}{l} A_{55}\overline{D}_{16}^2 I_1 + \overline{B}_{11}^2\overline{D}_{66} I_1 - A_{55}\overline{D}_{11}\overline{D}_{66} I_1 + \overline{A}_{11}\left(\overline{D}_{16}^2 - \overline{D}_{11}\overline{D}_{66}\right) I_1 + \\ 2A_{55}\overline{B}_{11}\overline{D}_{66} I_2 - 2\overline{B}_{11}\overline{D}_{66}\left(\overline{B}_{11} I_1 + A_{55} I_2\right) + \\ A_{55}\overline{B}_{11}^2 I_3 - \overline{A}_{11}A_{55}\left(\overline{D}_{11} + \overline{D}_{66}\right) I_3 + \overline{B}_{16}^2\left(\overline{D}_{11} I_1 + A_{55} I_3\right) \end{array}\right)\omega^2 \tag{22j}$$

$$\eta_2 = \omega^2\left(A_{55}\left(\overline{B}_{16}^2 - \overline{A}_{11}\overline{D}_{66}\right) I_1 - \left(\begin{array}{l} \overline{D}_{16}^2 I_1 - \overline{D}_{11}\overline{D}_{66} I_1 - 2\overline{B}_{16}\overline{D}_{16} I_1 I_2 + 2\overline{B}_{11}\overline{D}_{66} I_1 I_2 + A_{55}\overline{D}_{66} I_2^2 \\ + \left(\overline{B}_{11}^2 + \overline{B}_{16}^2 - \left(\overline{A}_{11} + A_{55}\right)\left(\overline{D}_{11} + \overline{D}_{66}\right)\right) I_1 + 2A_{55}\overline{B}_{11} I_2 \end{array}\right) I_3 - \overline{A}_{11}A_{55} I_3^2\right)\omega^2 \tag{22k}$$

$$\eta_1 = \omega^4\left(-A_{55} I_1\left(\overline{D}_{66} I_1 + \overline{A}_{11} I_3\right) + \left(\overline{D}_{66} I_1\left(-I_2^2 + I_1 I_3\right) + I_3\left(\begin{array}{l} \overline{D}_{11} I_1^2 - I_2\left(2\overline{B}_{11} I_1 + A_{55} I_2\right) + \\ \left(\overline{A}_{11} + A_{55}\right) I_1 I_3 \end{array}\right)\right)\omega^2\right) \tag{22l}$$

$$\eta_0 = I_1 I_3 \omega^6\left(-A_{55} I_1 + \left(-I_2^2 + I_1 I_3\right)\omega^2\right) \tag{22m}$$

Now the fourth order polynomial equation for the roots χ must be solved. Where $\chi=\kappa^2$ has substituted into equation (21) to reduce into a fourth order polynomial equation. The solutions can be found as follows-

$$\chi^4 + a_1\chi^3 + a_2\chi^2 + a_3\chi + a_4 = 0 \tag{23}$$

Where:-

$$\begin{aligned} a_1 &= \eta_3 / \eta_4 \\ a_2 &= \eta_2 / \eta_4 \\ a_3 &= \eta_1 / \eta_4 \\ a_4 &= \eta_0 / \eta_4 \end{aligned} \tag{24}$$

The fourth- order equation(23) can be factorized as-

$$\left(\chi^2 + p_1\chi + q_1\right)\left(\chi^2 + p_2\chi + q_2\right) = 0 \tag{25}$$

Where:-

$$\begin{aligned} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} &= \frac{1}{2}\left[a_1 \pm \sqrt{a_1^2 - 4a_2 + 4\lambda_1}\right] \\ \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= \frac{1}{2}\left[\lambda_1 \pm \frac{a_1\lambda_1 - 2a_3}{\sqrt{a_1^2 - 4a_2 + 4\lambda_1}}\right] \end{aligned} \tag{26}$$

And λ_1 is one of the roots of the following equation-

$$\lambda^3 - a_2\lambda^2 + \left(a_1a_3 - 4a_4\right)\lambda + \left(4a_2a_4 - a_3^2 - a_1^2a_4\right) = 0 \tag{27}$$

Then the roots of equation (23) can be written as-

$$\begin{aligned} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} &= \frac{-p_1}{2} \pm \sqrt{\frac{p_1^2}{4} - q_1} \\ \begin{pmatrix} \chi_3 \\ \chi_4 \end{pmatrix} &= \frac{-p_2}{2} \pm \sqrt{\frac{p_2^2}{4} - q_2} \end{aligned} \tag{28}$$

The roots of equation (27) can be written as-

$$\lambda_1 = \frac{a_2}{3} + 2\sqrt{-\bar{Q}}\cos(\vartheta/3)$$

$$\lambda_2 = \frac{a_2}{3} + 2\sqrt{-\bar{Q}}\cos((\vartheta + 2\pi)/3) \quad (29)$$

$$\lambda_3 = \frac{a_2}{3} + 2\sqrt{-\bar{Q}}\cos((\vartheta + 4\pi)/3)$$

Where:-

$$\vartheta = \cos^{-1}\left(\bar{R}/\sqrt{-\bar{Q}^3}\right)$$

$$\bar{Q} = -\frac{1}{9}\left(a_2^2 - 3a_1 a_3 + 12a_4\right)$$

$$\bar{R} = \frac{1}{54}\left(2a_2^3 - 9a_1 a_2 a_3 + 27a_3^2 + 27a_1^2 a_4 - 72a_2 a_4\right)$$

$$\bar{D} = \bar{Q}^3 + \bar{R}^2 \quad (30)$$

The general solutions to equations (19) are given by-

$$U(x) = A_1 e^{\kappa_1 x} + A_2 e^{-\kappa_1 x} + A_3 e^{\kappa_2 x} + A_4 e^{-\kappa_2 x} + A_5 e^{\kappa_3 x} + A_6 e^{-\kappa_3 x} + A_7 e^{\kappa_4 x} + A_8 e^{-\kappa_4 x}$$

$$= \sum_{j=1}^{4}\left(A_{2i-1}e^{\kappa_j x} + A_{2i} e^{-\kappa_j x}\right) \quad (31i)$$

$$W(x) = B_1 e^{\kappa_1 x} + B_2 e^{-\kappa_1 x} + B_3 e^{\kappa_2 x} + B_4 e^{-\kappa_2 x} + B_5 e^{\kappa_3 x} + B_6 e^{-\kappa_3 x} + B_7 e^{\kappa_4 x} + B_8 e^{-\kappa_4 x}$$

$$= \sum_{j=1}^{4}\left(B_{2i-1}e^{\kappa_j x} + B_{2i} e^{-\kappa_j x}\right) \quad (31j)$$

$$\Theta(x) = C_1 e^{\kappa_1 x} + C_2 e^{-\kappa_1 x} + C_3 e^{\kappa_2 x} + C_4 e^{-\kappa_2 x} + C_5 e^{\kappa_3 x} + C_6 e^{-\kappa_3 x} + C_7 e^{\kappa_4 x} + C_8 e^{-\kappa_4 x}$$

$$= \sum_{j=1}^{4}\left(C_{2i-1}e^{\kappa_j x} + C_{2i} e^{-\kappa_j x}\right) \quad (31k)$$

$$\Psi(x) = D_1 e^{\kappa_1 x} + D_2 e^{-\kappa_1 x} + D_3 e^{\kappa_2 x} + D_4 e^{-\kappa_2 x} + D_5 e^{\kappa_3 x} + D_6 e^{-\kappa_3 x} + D_7 e^{\kappa_4 x} + D_8 e^{-\kappa_4 x}$$

$$= \sum_{j=1}^{4}\left(D_{2i-1}e^{\kappa_j x} + D_{2i} e^{-\kappa_j x}\right) \quad (31l)$$

Where-

$\kappa_1 = \sqrt{\chi_1}$
$\kappa_2 = \sqrt{\chi_2}$
$\kappa_3 = \sqrt{\chi_3}$
$\kappa_4 = \sqrt{\chi_4}$

The relationship among the constants is given by-

$$A_{2j-1} = t_j C_{2j-1}$$

$$A_{2j} = t_j C_{2j} \qquad (32i)$$

$$B_{2j-1} = \hat{t}_j C_{2j-1}$$

$$B_{2j} = \hat{t}_j C_{2j} \qquad (32j)$$

$$D_{2j-1} = t'_j C_{2j-1}$$

$$D_{2j} = t'_j C_{2j} \qquad (32k)$$

Where:- (j=1-4)

$$t_j = \left[\bar{B}_{16}\bar{D}_{16}\kappa_j^4 - \left(\bar{B}_{11}\kappa_j^2 + I_2\omega^2\right)\left(\bar{D}_{66}\kappa_j^2 + I_3\omega^2\right)\right]/\Delta_j \qquad (33i)$$

$$\hat{t}_j = -A_{55}\kappa_j / \left(A_{55}\kappa_j^2 + I_1\omega^2\right) \qquad (33j)$$

$$t'_j = \kappa_j^2\left(\bar{B}_{11}\bar{B}_{16}\kappa_j^2 - \bar{A}_{11}\bar{D}_{16}\kappa_j^2 + \left(\bar{D}_{16}I_1 + \bar{B}_{16}I_2\right)\omega^2\right)/\Delta_j \qquad (33k)$$

$$\Delta_j = -\bar{B}_{16}^2\kappa_j^4 + \left(\bar{A}_{11}\kappa_j^2 + I_1\omega^2\right)\left(\bar{D}_{66}\kappa_j^2 + I_3\omega^2\right) \qquad (33l)$$

Sign convention:-

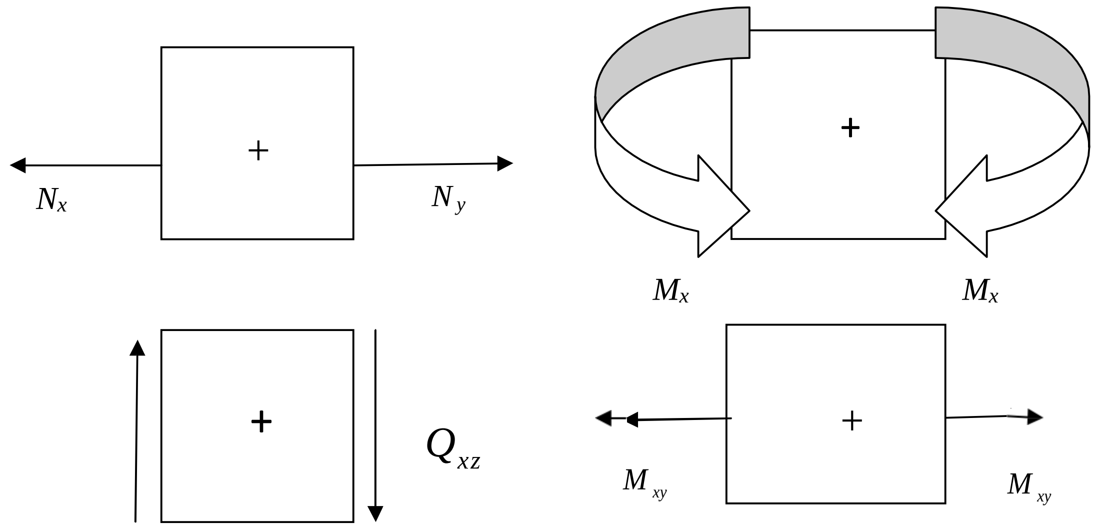

Fig. 2. Sign convention for positive normal force N_x (x), shear force Q_{xz} (x), bending moment M_x (x) and torque M_{xy} (x) [54]

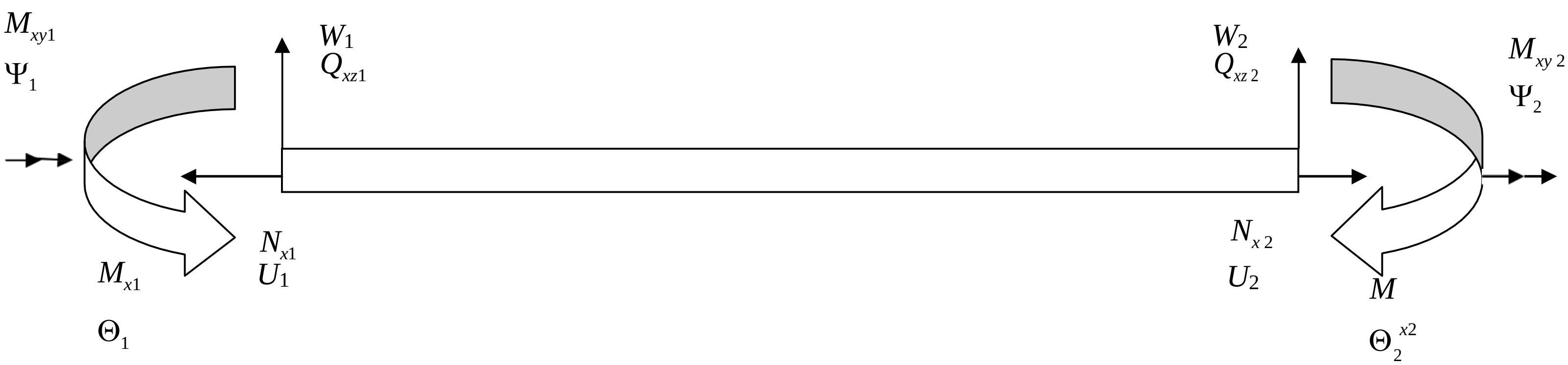

Fig.3. boundary conditions for displacements and forces of composite beam [54]

From fig. 2 the expression of normal force N_x (x), shear force Q_{xz} (x), bending moments M_x (x) and torque M_{xy} (x) can be obtained from equations (5), (6) and (31) as-

$$N_x(x) = A_{11}\frac{dU}{dx} + \overline{B_{11}}\frac{d\Theta}{dx} + \overline{B_{16}}\frac{d\Psi}{dx}$$

$$= \sum_{j=1}^{4}\left(A_{11}\kappa_j t_j + B_{11}\kappa_j + B_{16}\kappa_j t'_j\right)*\left(C_{2j-1}e^{\kappa_j x} - C_{2j}e^{-\kappa_j x}\right) \quad (34i)$$

$$Q_{xz}(x) = -\left(A_{55}\frac{dW}{dx} + A_{55}\Theta\right)$$

$$= \sum_{j=1}^{4}-\left(A_{55}\kappa_j t_j + A_{55}\right)*\left(C_{2j-1}e^{\kappa_j x} + C_{2j}e^{-\kappa_j x}\right) \quad (34j)$$

$$M_x(x) = -\left(B_{11}\frac{dU}{dx} + D_{11}\frac{d\Theta}{dx} + D_{16}\frac{d\Psi}{dx}\right)$$

$$= \sum_{j=1}^{4}-\left(B_{11}\kappa_j t_j + D_{11}\kappa_j + D_{16}\kappa_j t'_j\right)*\left(C_{2j-1}e^{\kappa_j x} - C_{2j}e^{-\kappa_j x}\right) \quad (34k)$$

$$M_{xy}(x) = \left(B_{16}\frac{dU}{dx} + D_{16}\frac{d\Theta}{dx} + D_{66}\frac{d\Psi}{dx}\right)$$

$$= \sum_{j=1}^{4}\left(B_{16}\kappa_j t_j + D_{16}\kappa_j + D_{66}\kappa_j t'_j\right)*\left(C_{2j-1}e^{\kappa_j x} - C_{2j}e^{-\kappa_j x}\right) \quad (34l)$$

From fig. 3. The boundary conditions for displacements and forces of the laminated composite beam are given as—

At:-

$x=0$;

$U = U_1$ $W = W_1$ $\Theta = \Theta_1$ $\Psi = \Psi_1$

$$N_x = -N_{x1} \quad Q_{xz} = Q_{xz1} \quad M_x = M_{x1} \quad M_{xy} = M_{xy1} \quad (35i)$$

At:-

x =L;

$$U = U_2 \qquad W = W_2 \qquad \Theta = \Theta_2 \qquad \Psi = \Psi_2$$

$$N_x = -N_{x2} \qquad Q_{xz} = Q_{xz2} \qquad M_x = M_{x2} \qquad M_{xy} = M_{xy2} \tag{35j}$$

Substituting equations (35) into equations (31), the nodal displacements defined by fig.(3) can be expressed in terms of C as-

$$\{D_0\} = [R]\{C\} \tag{36}$$

Where, $\{D_0\}$ is the nodal degree of freedom vector.

$$\{D_0\} = \{U_1 \quad W_1 \quad \Theta_1 \quad \Psi_1 \quad U_2 \quad W_2 \quad \Theta_2 \quad \Psi_2\} \tag{37i}$$

$$\{C\} = \{C_1 \quad C_3 \quad C_5 \quad C_7 \quad C_2 \quad C_4 \quad C_6 \quad C_8\} \tag{37j}$$

$$[R] = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 & t_1 & t_2 & t_3 & t_4 \\ \underline{t}_1 & \underline{t}_2 & \underline{t}_3 & \underline{t}_4 & -\underline{t}_1 & -\underline{t}_2 & -\underline{t}_3 & -\underline{t}_4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \tilde{t}_1 & t'_2 & t'_3 & t'_4 & \tilde{t}_1 & t'_2 & t'_3 & t'_4 \\ t_1 e^{\kappa_1 L} & t_2 e^{\kappa_2 L} & t_3 e^{\kappa_3 L} & t_4 e^{\kappa_4 L} & t_1 e^{-\kappa_1 L} & t_2 e^{-\kappa_2 L} & t_3 e^{-\kappa_3 L} & t_4 e^{-\kappa_4 L} \\ \bar{t}_1 e^{\kappa_1 L} & \bar{t}_2 e^{\kappa_2 L} & \bar{t}_3 e^{\kappa_3 L} & \bar{t}_4 e^{\kappa_4 L} & -\bar{t}_1 e^{-\kappa_1 L} & -\bar{t}_2 e^{-\kappa_2 L} & -\bar{t}_3 e^{-\kappa_3 L} & -\bar{t}_4 e^{-\kappa_4 L} \\ e^{\kappa_1 L} & e^{\kappa_2 L} & e^{\kappa_3 L} & e^{\kappa_4 L} & e^{-\kappa_1 L} & e^{-\kappa_2 L} & e^{-\kappa_3 L} & e^{-\kappa_4 L} \\ \tilde{t}_1 e^{\kappa_1 L} & t'_2 e^{\kappa_2 L} & t'_3 e^{\kappa_3 L} & t'_4 e^{\kappa_4 L} & \tilde{t}_1 e^{-\kappa_1 L} & t'_2 e^{-\kappa_2 L} & t'_3 e^{-\kappa_3 L} & t'_4 e^{-\kappa_4 L} \end{bmatrix} \tag{37k}$$

Substituting equations (35j) into equations (31), the nodal forces defined in fig. 3 can be expressed in terms of C as-

$$\{F_0\} = [H]\{C\} \tag{38}$$

Where, $\{F_0\}$ is the nodal force vector.

$$\{F_0\} = \{N_{x1} \quad Q_{xz1} \quad M_{x1} \quad M_{xy1} \quad N_{x2} \quad Q_{xz2} \quad M_{x2} \quad M_{xy2}\} \tag{39i}$$

$$[R] = \begin{bmatrix} -\hat{t}_1 & -\hat{t}_2 & -\hat{t}_3 & -\hat{t}_4 & \hat{t}_1 & \hat{t}_2 & \hat{t}_3 & \hat{t}_4 \\ \bar{\bar{t}}_1 & \bar{\bar{t}}_2 & \bar{\bar{t}}_3 & \bar{\bar{t}}_4 & \bar{\bar{t}}_1 & \bar{\bar{t}}_2 & \bar{\bar{t}}_3 & \bar{\bar{t}}_4 \\ \hat{\hat{t}}_1 & \hat{\hat{t}}_2 & \hat{\hat{t}}_3 & \hat{\hat{t}}_4 & -\hat{\hat{t}}_1 & -\hat{\hat{t}}_2 & -\hat{\hat{t}}_3 & -\hat{\hat{t}}_4 \\ -\tilde{t}_1 & -\tilde{t}_2 & -\tilde{t}_3 & -\tilde{t}_4 & \tilde{t}_1 & \tilde{t}_2 & \tilde{t}_3 & \tilde{t}_4 \\ \hat{t}_1 e^{\kappa_1 L} & \hat{t}_2 e^{\kappa_2 L} & \hat{t}_3 e^{\kappa_3 L} & \hat{t}_4 e^{\kappa_4 L} & -\hat{t}_1 e^{-\kappa_1 L} & -\hat{t}_2 e^{-\kappa_2 L} & -\hat{t}_3 e^{-\kappa_3 L} & -\hat{t}_4 e^{-\kappa_4 L} \\ -\bar{\bar{t}}_1 e^{\kappa_1 L} & -\bar{\bar{t}}_2 e^{\kappa_2 L} & -\bar{\bar{t}}_3 e^{\kappa_3 L} & -\bar{\bar{t}}_4 e^{\kappa_4 L} & -\bar{\bar{t}}_1 e^{-\kappa_1 L} & -\bar{\bar{t}}_2 e^{-\kappa_2 L} & -\bar{\bar{t}}_3 e^{-\kappa_3 L} & -\bar{\bar{t}}_4 e^{-\kappa_4 L} \\ -\hat{\hat{t}}_1 e^{\kappa_1 L} & -\hat{\hat{t}}_2 e^{\kappa_2 L} & -\hat{\hat{t}}_3 e^{\kappa_3 L} & -\hat{\hat{t}}_4 e^{\kappa_4 L} & \hat{\hat{t}}_1 e^{-\kappa_1 L} & \hat{\hat{t}}_2 e^{-\kappa_2 L} & \hat{\hat{t}}_3 e^{-\kappa_3 L} & \hat{\hat{t}}_4 e^{-\kappa_4 L} \\ \tilde{t}_1 e^{\kappa_1 L} & \tilde{t}_2 e^{\kappa_2 L} & \tilde{t}_3 e^{\kappa_3 L} & \tilde{t}_4 e^{\kappa_4 L} & -\tilde{t}_1 e^{-\kappa_1 L} & -\tilde{t}_2 e^{-\kappa_2 L} & -\tilde{t}_3 e^{-\kappa_3 L} & -\tilde{t}_4 e^{-\kappa_4 L} \end{bmatrix} \tag{39j}$$

Where, (j=1-6)

$$\hat{t}_j = \bar{A}_{11}\kappa_j t_j + \bar{B}_{11}\kappa_j + \bar{B}_{16}\kappa_j t'_j$$

$$\bar{\bar{t}}_j = -\left(A_{55}\kappa_j t_j + \bar{A}_{55}\right)$$

$$\hat{\hat{t}}_j = -\left(\overline{B_{11}}\kappa_j t_j + \overline{D_{11}}\kappa_j + \overline{D_{16}}\kappa_j t_j\right)'$$

$$\tilde{t}_j = \left(\overline{B_{16}}\kappa_j t_j + \overline{D_{16}}\kappa_j + \overline{D_{66}}\kappa_j t_j\right) \tag{40}$$

Now from the equations (36) and (38) relationship between the nodal force vector and nodal degree of freedom vector can be written as-

$$\{F_0\} = [H][R]^{-1}\{D_0\} = [K_0]\{D_0\} \quad (41)$$

Where,

$$[K_0] = [H][R]^{-1}$$

And $[K_0]$ is the frequency-dependent dynamic stiffness matrix.

HIGHER ORDER SHEAR DEFORMATION THEORY [55]

MATHAMATICAL FORMULATION

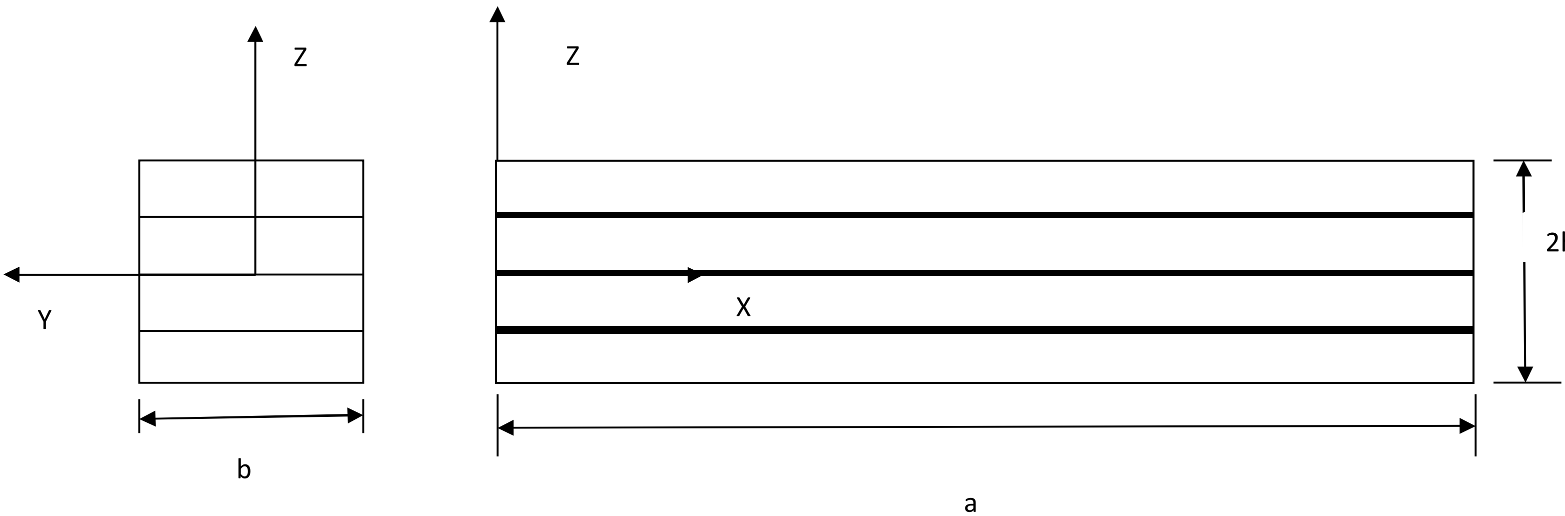

Fig.4. geometry and co-ordinate system of the beam [55]

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$u(x,z,t)$ and $w(x,z,t)$ are the displacement in the x and z directions, respectively, at any point (x,z).

h = half thickness of the beam,

φ_i = orientation angle of the i^{th} layer with respect to the x-axis.

a = length of the beam,

b = width of the beam.

The displacement fields for the first theory are taken as-

$$u = u_0 - z\frac{\partial w_{bn}}{\partial x} - \frac{z^2}{3h^2}\frac{\partial w_{sh}}{\partial x} - \frac{z^5}{5h^4}\frac{\partial w_2}{\partial x}$$

$$w = w_{bn} + w_{sh} + \left(\frac{z}{h}\right)^2 w_2 + \left[1 - \left(\frac{z}{h}\right)^4\right] w_4 \qquad (1)$$

The displacement fields for the second theory are taken as-

$$u = u_0 - z\frac{\partial w_{bn}}{\partial x} - \frac{z^2}{3h^2}\frac{\partial w_{sh}}{\partial x} - \frac{z^5}{5h^4}\frac{\partial w_2}{\partial x}$$

$$w = w_{bn} + w_{sh} + \left(\frac{z}{h}\right)^4 w_2 + \left[1 - \left(\frac{z}{h}\right)^2\right] w_4 \qquad (2)$$

Where:

$u_0(x, t)$ Are the displacements due to extension,

$w_{bn}(x,t)$ Are the displacements due to bending,

$w_{sh}(x,t)$ Are displacements due to shear deformation.

The terms $w_2(x, t)$ and $w_4(x, t)$ are the higher terms to include sectional wrapping and all these variables are measured at the mid surface of the beam z=0.

The displacement fields of the two theories are combined as follows:-

$$u = u_0 - z\frac{\partial w_{bn}}{\partial x} - \frac{z^3}{3h^2}\frac{\partial w_{sh}}{\partial x} - \frac{z^5}{5h^4}\frac{\partial w_2}{\partial x}$$

$$w = w_{bn} + w_{sh} + \left(\frac{z}{h}\right)^{2n+2} w_2 + (-1)^n\left[1 - \left(\frac{z}{h}\right)^{4-2n}\right] w_4 \qquad (3)$$

Where, n=0 for the first theory and n=1 for the second theory.

The strain – displacement relationship for these theories as-

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_{bn}}{\partial x^2} - \frac{z^3}{3h^2}\frac{\partial^2 w_{sh}}{\partial x^2} - \frac{z^5}{5h^4}\frac{\partial^2 w_2}{\partial x^2} \tag{4}$$

$$\varepsilon_{zz} = (2n+2)\frac{z^{2n+1}}{h^{2n+2}} w_2 - (-1)^n (4-2n)\frac{z^{3-2n}}{h^{4-2n}} w_4 \tag{5}$$

$$\varepsilon_{xz} = \left(1 - \frac{z^2}{h^2}\right)\frac{\partial w_{sh}}{\partial x} + \left[\left(\frac{z}{h}\right)^{2n+2} - \frac{z^4}{h^4}\right]\frac{\partial w_2}{\partial x} + (-1)^n \left[1 - \left(\frac{z}{h}\right)^{4-2n}\right]\frac{\partial w_4}{\partial x} \tag{6}$$

The strain fields are rewritten as-

$$\varepsilon_{xx} = \varepsilon_1^0 + z\varepsilon_1^1 + z^3\varepsilon_1^3 + z^5\varepsilon_1^5 \tag{7}$$

$$\varepsilon_{zz} = z^{2n+1}\varepsilon_3^1 + z^{3-2n}\varepsilon_3^3 \tag{8}$$

$$\varepsilon_{xz} = \varepsilon_4^0 + z^2\varepsilon_4^1 + z^{2n+2}\varepsilon_4^2 + z^4\varepsilon_4^3 + z^{4-2n}\varepsilon_4^4 \tag{9}$$

Where,

$$\varepsilon_1^0 = \frac{\partial u_0}{\partial x}$$

$$\varepsilon_1^1 = -\frac{\partial^2 w_{bn}}{\partial x^2}$$

$$\varepsilon_1^3 = -\frac{1}{3h^2}\frac{\partial^2 w_{sh}}{\partial x^2}$$

$$\varepsilon_1^5 = -\frac{1}{5h^4}\frac{\partial^2 w_2}{\partial x^2}$$

$$\varepsilon_3^1 = (2n+2)\frac{1}{h^{2n+2}} w_2$$

$$\varepsilon_3^3 = -(-1)^n (4-2n)\frac{1}{h^{4-2n}} w_4$$

$$\varepsilon_4^0 = \frac{\partial w_{sh}}{\partial x} + (-1)^n \frac{\partial w_4}{\partial x}$$

$$\varepsilon_4^1 = -\frac{1}{h^2}\frac{\partial w_{sh}}{\partial x} \qquad (10)$$

$$\varepsilon_4^2 = \left(\frac{1}{h}\right)^{2n+2}\frac{\partial w_2}{\partial x}$$

$$\varepsilon_4^3 = -\frac{1}{h^4}\frac{\partial w_2}{\partial x}$$

$$\varepsilon_4^4 = -(-1)^n \left[\left(\frac{1}{h}\right)^{4-2n}\right]\frac{\partial w_4}{\partial x}$$

The stress – strain relationship for the k^{th} layer is given as-

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \sigma_{xz} \end{Bmatrix}^k = \begin{bmatrix} Q_{11} & Q_{13} & 0 \\ Q_{13} & Q_{33} & 0 \\ 0 & 0 & Q_{44} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ \varepsilon_{xz} \end{Bmatrix}^k \qquad (11)$$

Where:-

Q_{11}, Q_{13}, Q_{33} and Q_{44} are the reduced material constants from the three dimensional orthotropic elasticity matrix.

By the use of Hamilton's principle, the equation of motion is given as-

$$\int_0^t \delta (T - U)\, dt = 0 \qquad (12)$$

Where, T is the kinetic energy and it is given as-

$$T = \frac{1}{2}\int_v \rho \left(\dot{u}^2 + \dot{w}^2\right) dV \qquad (13)$$

Where, $\dot{u}$ and $\dot{w}$ being time derivatives of u and w, and ρ is the density of the material.

U is the potential energy which is given as-

$$U = \frac{1}{2}\int_v \left(\zeta_{xx}\varepsilon_{xx} + \zeta_{zz}\varepsilon_{zz} + \zeta_{xz}\varepsilon_{xz}\right) dV \tag{14}$$

Substituting equation (13) and equation (14) into equation (12) –

$$\int_0 \int_v \left[\begin{array}{l} \left(\zeta_{xx}\delta\varepsilon_{xx} + \zeta_{zz}\delta\varepsilon_{zz} + \zeta_{xz}\delta\varepsilon_{xz}\right) - \rho\left(\frac{\partial u_0}{\partial t} - z\frac{\partial^2 w_{bn}}{\partial x\partial t} - \frac{z^3}{3h^2}\frac{\partial^2 w_{sh}}{\partial x\partial t} - \frac{z^5}{5h^4}\frac{\partial^2 w_2}{\partial x\partial t}\right) \\ \left(\frac{\partial \delta u_0}{\partial t} - z\frac{\partial^2 \delta w_{bn}}{\partial x\partial t} - \frac{z^3}{3h^2}\frac{\partial^2 \delta w_{sh}}{\partial x\partial t} - \frac{z^5}{5h^4}\frac{\partial^2 \delta w_2}{\partial x\partial t}\right) - \\ \rho\left(\frac{\partial w_{bn}}{\partial t} + \frac{\partial w_{sh}}{\partial t} + \left(\frac{z}{h}\right)^{2n+2}\frac{\partial w_2}{\partial t} + (-1)^n\left[1-\left(\frac{z}{h}\right)^{4-2n}\right]\frac{\partial w_4}{\partial t}\right) \\ \left(\frac{\partial \delta w_{bn}}{\partial t} + \frac{\partial \delta w_{sh}}{\partial t} + \left(\frac{z}{h}\right)^{2n+2}\frac{\partial \delta w_2}{\partial t} + (-1)^n\left[1-\left(\frac{z}{h}\right)^{4-2n}\right]\frac{\partial \delta w_4}{\partial t}\right) \end{array} \right] dVdt \tag{15}$$

Integrating the equation (15) by parts, and collecting the coefficients δw_{bn}, δw_{sh}, δw_2 and δw_4, the equation of motion in terms of stress resultant are given as-

$$\frac{\partial N}{\partial x} - \left(I_0\frac{\partial^2 u_0}{\partial t^2} - I_1\frac{\partial^3 w_{bn}}{\partial x\partial t^2} - \frac{I_3}{3h^2}\frac{\partial^3 w_{sh}}{\partial x\partial t^2} - \frac{I_5}{5h^4}\frac{\partial^3 w_2}{\partial x\partial t^2} \right) = 0 \tag{16a}$$

$$\frac{\partial^2 M}{\partial x^2} + \left(-I_1\frac{\partial^3 u_0}{\partial x\partial t^2} + I_2\frac{\partial^4 w_{bn}}{\partial x^2\partial t^2} + \frac{I_4}{3h^2}\frac{\partial^4 w_{sh}}{\partial x^2\partial t^2} + \frac{I_6}{5h^4}\frac{\partial^4 w_2}{\partial x^2\partial t^2} \right) - I_0\frac{\partial^2 w_{bn}}{\partial t^2} - I_0\frac{\partial^2 w_{sh}}{\partial t^2} - \frac{I_{2n+2}}{h^{2n+2}}\frac{\partial^2 w_2}{\partial t^2} - (-1)^n\left[I_0 - \frac{I_{4-2n}}{h^{4-2n}}\right]\frac{\partial^2 w_4}{\partial t^2} = 0 \tag{16b}$$

$$\frac{1}{3h^2}\frac{\partial^2 L}{\partial x^2} + \left(\frac{\partial Q_1}{\partial x} - \frac{1}{h^2}\frac{\partial Q_2}{\partial x}\right) + \left(\frac{-I_3}{3h^2}\frac{\partial^3 u_0}{\partial x\partial t^2} + \frac{I_4}{3h^2}\frac{\partial^4 w_{bn}}{\partial x^2\partial t^2} + \frac{I_6}{9h^4}\frac{\partial^4 w_{sh}}{\partial x^2\partial t^2} + \frac{I_8}{15h^6}\frac{\partial^4 w_2}{\partial x^2\partial t^2} \right) - \left[I_0\frac{\partial^2 w_{bn}}{\partial t^2} + I_0\frac{\partial^2 w_{sh}}{\partial t^2} + \frac{I_{2n+2}}{h^{2n+2}}\frac{\partial^2 w_2}{\partial t^2} + (-1)^n\left[I_0 - \frac{I_{4-2n}}{h^{4-2n}}\right]\frac{\partial^2 w_4}{\partial t^2} \right] = 0 \tag{16c}$$

$$\frac{1}{3h^2}\frac{\partial^2 P}{\partial x^2} - \frac{2n+2}{h^{2n+2}}V_1 + \left(\frac{1}{h^{2n+2}}\frac{\partial Q_3}{\partial x} - \frac{1}{h^4}\frac{\partial Q_4}{\partial x}\right) + \left(\frac{-I_5}{5h^6}\frac{\partial^3 u_0}{\partial x\partial t^2} + \frac{I_6}{5h^4}\frac{\partial^4 w_{bn}}{\partial x^2\partial t^2} + \frac{I_8}{15h^6}\frac{\partial^4 w_{sh}}{\partial x^2\partial t^2} + \frac{I_{10}}{25h^8}\frac{\partial^4 w_2}{\partial x^2\partial t^2}\right)$$
$$-\left[\frac{I_{2n+2}}{h^{2n+2}}\frac{\partial^2 w_{bn}}{\partial t^2} + \frac{I_{2n+2}}{h^{2n+2}}\frac{\partial^2 w_{sh}}{\partial t^2} + \frac{I_{4n+4}}{h^{4n+4}}\frac{\partial^2 w_2}{\partial t^2} + (-1)^n\left[\frac{I_{2n+2}}{h^{2n+2}} - \frac{I_6}{h^6}\right]\frac{\partial^2 w_4}{\partial t^2}\right] = 0$$

(16d)

$$\frac{4-2n}{h^{4-2n}}V_2 + \left(\frac{\partial Q_1}{\partial x} - \frac{1}{h^{4-2n}}\frac{\partial Q_5}{\partial x}\right) - \left(I_0 - \frac{I_{4-2n}}{h^{4-2n}}\right)\frac{\partial^2 w_{bn}}{\partial t^2} - \left(I_0 - \frac{I_{4-2n}}{h^{4-2n}}\right)\frac{\partial^2 w_{sh}}{\partial t^2}$$
$$-\left(\frac{I_{2n+2}}{h^{2n+2}} - \frac{I_6}{h^6}\right)\frac{\partial^2 w_2}{\partial t^2} - (-1)^n\left[I_0 - \frac{2I_{4-2n}}{h^{4-2n}} + \frac{I_{8-4n}}{h^{8-4n}}\right]\frac{\partial^2 w_4}{\partial t^2} = 0$$

(16e)

The laminate stiffness constants are given as-

$$\left(A_{11}, A_{12}, A_{13}, A_{14}\right) = \sum_{i=1}^{NL}\int_{z_i}^{z_i+1} Q_{11}^i\left(1, z, z^3, z^5\right)dz$$

$$\left(A_{15}, A_{16}\right) = \sum_{i=1}^{NL}\int_{z_i}^{z_i+1} Q_{13}^i\left(z^{2n+1}, z^{3-2n}\right)dz$$

$$\left(B_{11}, B_{12}, B_{13}\right) = \sum_{i=1}^{NL}\int_{z_i}^{z_i+1} Q_{11}^i\left(z^2, z^4, z^6\right)dz$$

$$\left(B_{14}, B_{15}\right) = \sum_{i=1}^{NL}\int_{z_i}^{z_i+1} Q_{13}^i\left(z^{2n+2}, z^{4-2n}\right)dz$$

$$\left(C_{11}\right) = \sum_{i=1}^{NL}\int_{z_i}^{z_i+1} Q_{11}^i\left(z^8\right)dz$$

$$\left(C_{12}, C_{13}\right) = \sum_{i=1}^{NL}\int_{z_i}^{z_i+1} Q_{13}^i\left(z^{2n+2}, z^{6-2n}\right)dz$$

$$\left(D_{11}\right) = \sum_{i=1}^{NL}\int_{z_i}^{z_i+1} Q_{11}^i\left(z^{10}\right)dz$$

$$\left(D_{12}, D_{13}\right) = \sum_{i=1}^{NL}\int_{z_i}^{z_i+1} Q_{13}^i\left(z^{2n+6}, z^{8-2n}\right)dz$$

$$\left(F_{11}\right)=\sum_{i=1}^{NL}\int_{z_i}^{z_i+1} Q_{33}^{i}\left(z^{6-4n}\right)dz$$

$$\left(E_{11},E_{12}\right)=\sum_{i=1}^{NL}\int_{z_i}^{z_i+1} Q_{33}^{i}\left(z^{4n+2},z^{4}\right)dz$$

$$\left(G_{11},G_{12},G_{13},G_{14},G_{15}\right)=\sum_{i=1}^{NL}\int_{z_i}^{z_i+1} Q_{44}^{i}\left(1,z,z^{2n+2},z^{4},z^{4-2n}\right)dz$$

$$\left(H_{11},H_{12},H_{13}\right)=\sum_{i=1}^{NL}\int_{z_i}^{z_i+1} Q_{44}^{i}\left(z^{2n+4},z^{6},z^{6-2n}\right)dz$$

$$\left(T_{11}\right)=\sum_{i=1}^{NL}\int_{z_i}^{z_i+1} Q_{44}^{i}\left(z^{8-4n}\right)dz$$

$$\left(R_{11},R_{12}\right)=\sum_{i=1}^{NL}\int_{z_i}^{z_i+1} Q_{44}^{i}\left(z^{4n+4},z^{2n+6}\right)dz$$

$$\left(S_{11},S_{12}\right)=\sum_{i=1}^{NL}\int_{z_i}^{z_i+1} Q_{44}^{i}\left(z^{8},z^{8-2n}\right)dz$$

NL is the number of layers. The stress resultant defined as –

$$\left(N,M,L,P\right)=\sum_{i=1}^{NL}\int_{z_i}^{z_i+1}\zeta_{xx}\left(1,z,z^{3},z^{5}\right)dz$$

$$\left(V_1,V_2\right)=\sum_{i=1}^{NL}\int_{z_i}^{z_i+1}\zeta_{zz}\left(z^{2n+1},z^{3-2n}\right)dz$$

$$\left(Q_1,Q_2,Q_3,Q_4,Q_5\right)=\sum_{i=1}^{NL}\int_{z_i}^{z_i+1}\zeta_{xz}\left(1,z,z^{3},z^{5},z^{7}\right)dz$$

The mass moment of inertia given as-

$$\left(I_0,I_1,I_3,I_4,I_5,I_6,I_{2n+2},I_{4-2n},I_8,I_{10},I_{4n+4},I_{8-4n}\right)=$$
$$\sum_{i=1}^{NL}\int_{z_i}^{z_i+1}\rho\left(1,z,z^{2},z^{3},z^{4},z^{5},z^{6},z^{2n+2},z^{4n-2},z^{8},z^{10},z^{4n+4},z^{8-4n}\right)dz \tag{17}$$

The equation of motion are expressed in terms of the displacement as follows-

$$A_{11}\frac{\partial^2 u_0}{\partial x^2} - A_{12}\frac{\partial^3 w_{bn}}{\partial x^3} - \frac{A_{13}}{3h^2}\frac{\partial^3 w_{sh}}{\partial x^3} - \frac{A_{14}}{5h^4}\frac{\partial^3 w_2}{\partial x^3} + \frac{2n+2}{h^{2n+2}}A_{15}\frac{\partial w_2}{\partial x} - (-1)^n\left[\frac{4-2n}{h^{4-2n}}\right]A_{16}\frac{\partial w_4}{\partial x} - \left(I_0\frac{\partial^2 u_0}{\partial t^2} - I_1\frac{\partial^3 w_{bn}}{\partial x\partial t^2} - \frac{I_3}{3h^2}\frac{\partial^3 w_{sh}}{\partial x\partial t^2} - \frac{I_5}{5h^4}\frac{\partial^3 w_2}{\partial x\partial t^2}\right) = 0 \qquad (18a)$$

$$A_{12}\frac{\partial^3 u_0}{\partial x^3} - B_{11}\frac{\partial^4 w_{bn}}{\partial x^4} - \frac{B_{12}}{3h^2}\frac{\partial^4 w_{sh}}{\partial x^4} - \frac{B_{13}}{5h^4}\frac{\partial^4 w_2}{\partial x^4} + \frac{2n+2}{h^{2n+2}}B_{14}\frac{\partial^2 w_2}{\partial x^2} - (-1)^n\left[\frac{4-2n}{h^{4-2n}}\right]B_{15}\frac{\partial^2 w_4}{\partial x^2} - I_0\frac{\partial^2 w_{bn}}{\partial t^2} - I_0\frac{\partial^2 w_{sh}}{\partial t^2} - \left[\frac{I_{2n+2}}{h^{2n+2}}\right]\frac{\partial^2 w_2}{\partial t^2} - (-1)^n\left(I_0 - \frac{I_{4-2n}}{h^{4-2n}}\right)\frac{\partial^2 w_4}{\partial t^2} - I_1\frac{\partial^3 u_0}{\partial x\partial t^2} + I_2\frac{\partial^4 w_{bn}}{\partial x^2\partial t^2} + \frac{I_4}{3h^2}\frac{\partial^4 w_{sh}}{\partial x^2\partial t^2} + \frac{I_6}{5h^4}\frac{\partial^4 w_2}{\partial x^2\partial t^2} = 0$$

(18b)

$$\frac{A_{13}}{3h^2}\frac{\partial^3 u_0}{\partial x^3} - \frac{B_{12}}{3h^2}\frac{\partial^4 w_{bn}}{\partial x^4} - \frac{B_{13}}{9h^4}\frac{\partial^4 w_{sh}}{\partial x^4} - \frac{C_{11}}{15h^6}\frac{\partial^4 w_2}{\partial x^4} + \left(G_{11} - \frac{2G_{12}}{h^2} - \frac{G_{14}}{h^4}\right)\frac{\partial^2 w_{sh}}{\partial x^2} + \left(\frac{2n+2}{3h^{2n+2}}C_{12} + \frac{G_{13}}{h^{2n+2}} - \frac{G_{14}}{h^4} - \frac{H_{11}}{h^{2n+4}} + \frac{H_{12}}{h^6}\right)\frac{\partial^2 w_2}{\partial x^2} + \left(-\frac{I_3}{3h^2}\frac{\partial^3 u_0}{\partial x\partial t^2} + \frac{I_4}{3h^2}\frac{\partial^4 w_{bn}}{\partial x^2\partial t^2} + \frac{I_6}{9h^4}\frac{\partial^4 w_{sh}}{\partial x^2\partial t^2} + \frac{I_8}{15h^6}\frac{\partial^4 w_2}{\partial x^2\partial t^2}\right) - \left[I_0\frac{\partial^2 w_{bn}}{\partial t^2} + I_0\frac{\partial^2 w_{sh}}{\partial t^2} + \left[\frac{I_{2n+2}}{h^{2n+2}}\right]\frac{\partial^2 w_2}{\partial t^2} + (-1)^n\left(I_0 - \frac{I_{4-2n}}{h^{4-2n}}\right)\frac{\partial^2 w_4}{\partial t^2}\right] = 0$$

(18c)

$$\frac{A_{14}}{5h^4}\frac{\partial^3 u_0}{\partial x^3} - \frac{B_{13}}{5h^4}\frac{\partial^4 w_{bn}}{\partial x^4} - \frac{C_{11}}{15h^6}\frac{\partial^4 w_{sh}}{\partial x^4} - \frac{D_{11}}{25h^8}\frac{\partial^4 w_2}{\partial x^4} - \frac{2n+2}{h^{2n+2}}A_{15}\frac{\partial u_0}{\partial x} + \frac{2n+2}{h^{2n+2}}B_{14}\frac{\partial^2 w_{bn}}{\partial x^2} + \left(\frac{2n+2}{3h^{2n+4}}C_{12} + \frac{G_{13}}{h^{2n+2}} - \frac{G_{14}}{h^4} - \frac{H_{11}}{h^{2n+4}} + \frac{H_{12}}{h^6}\right)\frac{\partial^2 w_{sh}}{\partial x^2} + \left[\frac{4n+4}{5h^{2n+6}}D_{12} + \frac{R_{11}}{h^{4n+4}} - \frac{2R_{12}}{h^{2n+6}} + \frac{S_{11}}{h^8}\right]\frac{\partial^2 w_2}{\partial x^2}$$

$$+(-1)^n\left[\frac{4-2n}{5h^{8-2n}}D_{13} + \frac{G_{13}}{h^{2n+2}} - \frac{H_{12}}{h^6} - \frac{G_{14}}{h^4} + \frac{S_{12}}{h^{8-2n}}\right]\frac{\partial^2 w_4}{\partial x^2} - \frac{2n+2}{h^{2n+2}}E_{11}w_2 + (-1)^n\frac{(2n+2)(4-2n)}{h^6}E_{12}w_4 + \left(-\frac{I_5}{5h^6}\frac{\partial^3 u_0}{\partial x\partial t^2} + \frac{I_6}{5h^4}\frac{\partial^4 w_{bn}}{\partial x^2\partial t^2} + \frac{I_8}{15h^6}\frac{\partial^4 w_{sh}}{\partial x^2\partial t^2} + \frac{I_{10}}{25h^8}\frac{\partial^4 w_2}{\partial x^2\partial t^2}\right) - \left[\frac{I_{2n+2}}{h^{2n+2}}\frac{\partial^2 w_{bn}}{\partial t^2} + \frac{I_{2n+2}}{h^{2n+2}}\frac{\partial^2 w_0}{\partial t^2} + \left[\frac{I_{4n+4}}{h^{4n+4}}\right]\frac{\partial^2 w_2}{\partial t^2}\right] + (-1)^n\left(\frac{I_{2n+2}}{h^{2n+2}} - \frac{I_6}{h^6}\right)\frac{\partial^2 w_4}{\partial t^2} = 0$$

(18d)

$$\frac{4-2n}{h^{4-2n}}A_{16}\frac{\partial u_0}{\partial x}-\frac{4-2n}{h^{4-2n}}B_{15}\frac{\partial^2 w_{bn}}{\partial x^2}+\left[\frac{4-2n}{h^{4-2n}}\frac{C_{13}}{3h^2}+G_{11}-\frac{G_{12}}{h^2}-\frac{G_{15}}{h^{4-2n}}-\frac{H_{13}}{h^{6-2n}}\right]\frac{\partial^2 w_{sh}}{\partial x^2}+\left[\frac{4-2n}{h^{4-2n}}\frac{D_{13}}{5h^4}+\frac{G_{13}}{h^{2n+2}}-\frac{G_{14}}{h^4}-\frac{H_{12}}{h^6}+\frac{H_{12}}{h^{8-2n}}\right]\frac{\partial^2 w_2}{\partial x^2}+(-1)^n\left[\left(G_{11}-\frac{G_{15}}{h^{4-2n}}\right)-(-1)^n\frac{I}{h^{4-2n}}\left(G_{15}-\frac{T_{11}}{h^{4-2n}}\right)\right]\frac{\partial^2 w_4}{\partial x^2}$$

$$+\frac{(2n+2)(4-2n)}{h^6}E_{12}w_2-(-1)^n\left(\frac{4-2n}{h^{4-2n}}\right)^2F_{11}w_4-\left(I_0-\frac{I_{4-2n}}{h^{4-2n}}\right)\frac{\partial^2 w_{sh}}{\partial t^2}-\left(I_0-\frac{I_{4-2n}}{h^{4-2n}}\right)\frac{\partial^2 w_{bn}}{\partial t^2}-$$

$$\left(\frac{I_{2n+2}}{h^{2n+2}}-\frac{I_6}{h^6}\right)\frac{\partial^2 w_2}{\partial x^2}-(-1)^n\left(I_0-\frac{2I_{4-2n}}{h^{4-2n}}-\frac{I_{8-4n}}{h^{8-4n}}\right)\frac{\partial^2 w_4}{\partial x^2}=0 \quad (18e)$$

Analytical solution of the equation of motion [55]

Closed form solution for the above equations can be obtained for a simply supported beam by assuming –

$$u_0=\sum A(t)\cos px,\ (w_{bn},w_{sh},w_2,w_4)$$

$$=\sum\big(B(t),C(t),D(t),E(t)\big)\sin px \quad (19)$$

Where, $p=\frac{n\pi}{a}$

After substitution equation (19) into equation (18) , the following sets of equations is obtained-

$$[K]\{X\}+[M]\{\ddot{X}\}=0 \quad (20)$$

Now the elements of the stiffness matrix [K] are given as-

$K_{11}=-p^2A_{11}$, $K_{12}=p^3A_{12}$, $K_{13}=p^3\frac{A_{13}}{3h^2}$, $K_{14}=p^3\frac{A_{14}}{5h^4}-p\frac{2n+2}{h^{2n+2}}A_{15}$

$K_{15}=(-1)^n\left[\frac{4-2n}{h^{4-2n}}\right]A_{16}$, $K_{22}=-p^4B_{11}$, $K_{23}=-p^4\frac{B_{12}}{3h^2}$, $K_{24}=-p^4\frac{B_{13}}{5h^4}-p^2\frac{2n+2}{h^{2n+2}}A_{14}$

$K_{25}=(-1)^n p^2\left[\frac{4-2n}{h^{4-2n}}\right]B_{15}$, $K_{33}=-p^4\frac{B_{13}}{9h^4}-p^2\left(G_{11}-\frac{2G_{12}}{h^2}-\frac{G_{14}}{h^4}\right)$,

$$K_{34} = -p^4 \frac{C_{11}}{15h^6} - p^2 \left(\frac{2n+2}{3h^{2n+2}} C_{12} + \frac{G_{13}}{h^{2n+2}} - \frac{G_{14}}{h^4} - \frac{H_{11}}{h^{2n+4}} + \frac{H_{12}}{h^6} \right),$$

$$K_{35} = -(-1)^n p^2 \left[-\frac{4-2n}{h^{4-2n}} \frac{C_{13}}{3h^2} + G_{11} - \frac{G_{12}}{h^2} - \frac{G_{15}}{h^{4-2n}} - \frac{H_{13}}{h^{6-2n}} \right],$$

$$K_{44} = -p^4 \frac{D_{11}}{25h^8} - p^2 \left[\frac{4n+4}{5h^{2n+6}} D_{12} + \frac{R_{11}}{h^{4n+4}} - \frac{2R_{12}}{h^{2n+6}} + \frac{S_{11}}{h^8} \right] - \frac{2n+2}{h^{2n+2}} E_{11},$$

$$K_{45} = -p^2 (-1)^n \left[\frac{4-2n}{5h^{8-2n}} D_{13} + \frac{G_{13}}{h^{2n+2}} - \frac{H_{12}}{h^6} - \frac{G_{14}}{h^4} + \frac{S_{12}}{h^{8-2n}} \right] + (-1)^n \frac{(2n+2)(4-2n)}{h^6} E_{12},$$

$$K_{55} = -p^2 \left[(-1)^n \left(G_{11} - \frac{G_{15}}{h^{4-2n}} \right) - (-1)^n \frac{1}{h^{4-2n}} \left(G_{15} - \frac{T_{11}}{h^{4-2n}} \right) \right] - (-1)^n \left(\frac{(4-2n)}{h^{4-2n}} \right)^2 F_{11}.$$

And the mass matrix written as-

$$[M] = \begin{bmatrix} -I_0 & pI_1 & p\frac{I_3}{3h^2} & p\frac{I_5}{5h^4} & 0 \\ pI_1 & -(-I_0 + p^2 I_2) & -\left(I_0 + p^2 \frac{I_4}{3h^2}\right) & -\left(\frac{I_{2n+2}}{h^{2n+2}} + p^2 \frac{I_6}{5h^4}\right) & -(-1)^n \left(I_0 - \frac{I_{4-2n}}{h^{4-2n}}\right) \\ p\frac{I_3}{3h^2} & -\left(I_0 + p^2 \frac{I_4}{3h^2}\right) & -\left(I_0 + p^2 \frac{I_6}{9h^4}\right) & -\left(\frac{I_{2n+2}}{h^{2n+2}} + p^2 \frac{I_8}{15h^6}\right) & -(-1)^n \left(I_0 - \frac{I_{4-2n}}{h^{4-2n}}\right) \\ p\frac{I_5}{5h^4} & -\left(\frac{I_{2n+2}}{h^{2n+2}} + p^2 \frac{I_6}{5h^4}\right) & -\left(\frac{I_{2n+2}}{h^{2n+2}} + p^2 \frac{I_8}{15h^6}\right) & \left(\frac{I_{4n+4}}{h^{4n+4}} - p^2 \frac{I_8}{15h^6} \frac{I_{10}}{25h^8}\right) & -(-1)^n \left(\frac{I_{2n+2}}{h^{2n+2}} - p^2 \frac{I_{10}}{25h^8}\right) \\ 0 & -(-1)^n \left(I_0 - \frac{I_{4-2n}}{h^{4-2n}}\right) & -(-1)^n \left(I_0 - \frac{I_{4-2n}}{h^{4-2n}}\right) & -(-1)^n \left(\frac{I_{2n+2}}{h^{2n+2}} - p^2 \frac{I_{10}}{25h^8}\right) & -(-1)^n \left(I_0 - \frac{2I_{4-2n}}{h^{4-2n}} - \frac{I_{8-4n}}{h^{8-4n}}\right) \end{bmatrix}$$

And the displacement vector are expressed as-

$$\{X\} = \{A \quad B \quad C \quad D \quad E\}$$

$$\{X\} = \{A \quad B \quad C \quad D \quad E\}$$

It is assumed that to find the solution of equation (20)

$$\{A \quad B \quad C \quad D \quad E\} = \{A_0 \quad B_0 \quad C_0 \quad D_0 \quad E_0\} e^{i\omega_n t}$$

Where, A_0 B_0 C_0 D_0 E_0 are the constants, and ω_n is the natural frequency.

By substituting the values for A, B, C, D and E into equation (20) the natural frequencies and the corresponding mode shapes can be obtained [54].

Finite element formulation[54]

For the present analysis we assumed, each element having two nodes and each node having eight degrees of freedom. Linear polynomials are used for the nodal variables u_0 and w_4 and hermite cubic polynomials are used for the other nodal variables of the elements.

The displacements fields given are rewritten in the matrix form as-

$$\{U_d\} = [Z_d]\{d\} \tag{21}$$

$$\{U_d\} = \{u \quad w\}^T \tag{22}$$

$$[Z_d] = \begin{bmatrix} 1 & 0 & z & 0 & -\dfrac{z^3}{3h^2} & 0 & -\dfrac{z^5}{5h^4} & 0 \\ 0 & 1 & 0 & 1 & 0 & \left(\dfrac{z}{h}\right)^{2n+2} & 0 & (-1)^n\left(1-\left(\dfrac{z}{h}\right)^{4-2n}\right) \end{bmatrix} \tag{23}$$

$$\{d\} = \left\{u_0 \quad w_{bn} \quad \frac{\partial w_{bn}}{\partial x} \quad w_{sh} \quad \frac{\partial w_{sh}}{\partial x} \quad w_2 \quad \frac{\partial w_2}{\partial x} \quad w_4\right\} \tag{24}$$

The strain field associated with equation (21) is given as-

$$\{\varepsilon\} = [Z_{id}]\{\kappa\} \tag{25}$$

$$\{\varepsilon\} = \{\varepsilon_{xx} \quad \varepsilon_{zz} \quad \varepsilon_{xz}\}^T \tag{26}$$

$$[Z_{id}] = \begin{bmatrix} 1 & z & z^5 & z^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z^{2n+1} & z^3-2n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & z^2 & z^{2n+2} & z^4 & z^{4-2n} \end{bmatrix} \tag{27}$$

$$\{\kappa\} = \{\varepsilon_1^0\ \varepsilon_1^1\ \varepsilon_1^3\ \varepsilon_1^5\ \varepsilon_1^1\ \varepsilon_1^3\ \varepsilon_3^0\ \varepsilon_3^1\ \varepsilon_4^2\ \varepsilon_4^3\ \varepsilon_4^4\}^T \tag{28}$$

From equation (18) we get-

$$\begin{aligned} \{\delta U_d\} &= \int_v \{\delta\varepsilon_h\}^T \{\zeta\}\, dV \\ &= b\int_0^x \int_{-h}^{h} \{\delta\kappa\}^T [Z_{id}]^T \{\zeta\}\, dz dA \\ &= b\int_0^x \{\delta\kappa\}^T \left(\int_{-h}^{h} [Z_{id}]^T \{\zeta\}\, dz \right) dx \\ &= b\int_0^x \{\delta\kappa\}^T \{S_d\}\, dx \end{aligned} \tag{29}$$

Where the stress resultant $\{S_d\}$ given as- $\{S_d\} = \{N\ M\ L\ V_1\ V_2\ Q_L\ Q_2\ Q_3\ Q_4\ Q_6\}$

$$\begin{aligned} \{S_d\} &= \int_{-h}^{h} [Z_{id}]^T \{\zeta\}\, dz \\ &= \sum_{i=1}^{NL} \int_{z_i}^{z_{i+1}} [Z_{id}]^T [Q][Z_{id}]\{\kappa\}\, dz \\ &= [D_0]\{\kappa\} \end{aligned} \tag{30}$$

Where,

$$[D_0] = \sum_{i=1}^{NL} \int_{z_i}^{z_{i+1}} [Z_{id}]^T [Q][Z_{id}]\, dz$$

Now substituting equation (30) into equation (29)-

$$\delta U_d = b\int_0^x \{\delta\kappa\}^T [D_0]\{S_d\}\, dx \tag{31}$$

From equation (13)-

$$\int_0^t \int_v \delta T\, dV dt = \int_0^t \int_v \rho \left(\dot{u}\delta\dot{u} + \dot{w}\delta\dot{w} \right) dV dt$$
$$= -\rho \int_0^t \int_v \{\delta U_d\}^T \{\ddot{U}_d\} dV dt$$
$$= -\rho \int_0^t \int_v \{\delta d\}^T [Z_{id}]^T [Z_{id}] \{\ddot{d}\} dV dt \tag{32}$$
$$= -\rho b \int_0^t \int_0^x \{\delta d\}^T \left(\int_{z_i}^{z_{i+1}} [Z_{id}]^T [Z_{id}] dz \right) \{\ddot{d}\} dx dt$$
$$= -b \int_0^t \int_0^x \{\delta d\}^T [I_{moi}] \{\ddot{d}\} dx dt$$

Here the mass moment of inertia matrix –

$$[I_{moi}] = \rho \int_{z_i}^{z_{i+1}} [Z_{id}]^T [Z_{id}] dz \tag{33}$$

Now substituting equations (31) and (32) into Hamilton's principle, the expression for the equation of motion is given as-

$$\rho \int_0^t \int_0^x \{\delta d\}^T [I_{moi}] \{\ddot{d}\} dx dt + b \int_0^t \int_0^x \{\delta \kappa\}^T [D_0] \{\kappa\} dx dt = 0 \tag{34}$$

Derivation of stiffness and consistent mass matrices [55]

The displacement vector within an element can be expressed in terms of the nodal degree of freedom as-

$$\{d\} = \{[\partial][N]\}\{\delta\} \tag{35}$$

Where,

$$[\partial] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

And the shape function are taken as-

$$[N] = \begin{bmatrix} N_1^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_2^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & N_1 & N_2 & 0 & 0 & 0 & 0 & 0 & 0 & N_3 & N_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & 0 & 0 & 0 & 0 & 0 & 0 & N_3 & N_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & N_1 & N_2 & 0 & 0 & 0 & 0 & 0 & 0 & N_3 & N_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_1^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_2^1 \end{bmatrix}$$

And the nodal degree of freedom given by-

$$\{\underline{\delta}\} = \left\{ u_{0_1}\quad w_{bn_2}\quad \left(\frac{\partial w_{bn}}{\partial t}\right)_3\quad w_{sh_2}\quad \left(\frac{\partial w_{sh}}{\partial t}\right)_5\ \ldots\ldots\ldots w_{2_{14}}\quad \left(\frac{\partial w_2}{\partial t}\right)_{15}\quad w_{16} \right\}$$

The strain curvature vector within an element is given as-

$$\{\kappa\} = [B]\{\delta\}$$

Where,

$$[B] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3h^2}\frac{\partial^2}{\partial x^2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{5h^4}\frac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 & 0 & \frac{2n+2}{h^{2n+2}} & 0 \\ 0 & 0 & 0 & 0 & \frac{(-1)^{n+1}(4-2n)}{h^{4-2n}} \\ 0 & 0 & \frac{\partial}{\partial x} & 0 & (-1)^n \frac{\partial}{\partial x} \\ 0 & 0 & -\frac{1}{h^2}\frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{h^{2n+2}}\frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & -\frac{1}{h^4}\frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & \frac{(-1)^{n+1}}{h^{4-2n}}\frac{\partial}{\partial x} \end{bmatrix} [N] \qquad (36)$$

Equations of motion [54]

To derive equation of motion using Hamilton's principle, equation (36) and (35) are substituted into equation (34)-

$$\int_0^t\int_0^x \left(\delta\{\dot{\delta}\}^T \{[\partial][N]\}^T [I_{moi}]^T \{[\partial][N]\}\{\ddot{\delta}\}\right) dxdt + b\int_0^t\int_0^x \left(\delta\{\delta\}^T [B]^T [D_0][B]\{\delta\}\right) dxdt = 0 \quad (37)$$

And also we can write:-

$$[M]^{el}\{\ddot{\delta}\}^{el} + [K]^{el}\{\delta\}^{el} = 0$$

Where the element mass matrix and stiffness matrix are given as-

$$[M]^{el} = b\sum_{i=1}^{NL}\int \{[\partial][N]\}^T [I_{moi}]^T \{[\partial][N]\} dx$$

$$[K]^{el} = b\sum_{i=1}^{NL}\int [B]^T [D_0][B] dx$$

Non dimensional natural frequency $= \bar{\omega} = \omega_n a^2 \left[\rho / \left(4E_1 h^2\right)\right]^{1/2}$.

Chapter 4

COMPUTER PROGRAM

COMPUTER PROGRAM FOR THE LAMINATED COMPOSITE BEAM

Boundary Condition

The material properties are assigned to the beam and boundary conditions are defined. The beam‘s all degrees of freedom on surface are taken. They are denoted with the blue flag. This condition prevents the movement of the surface in a space

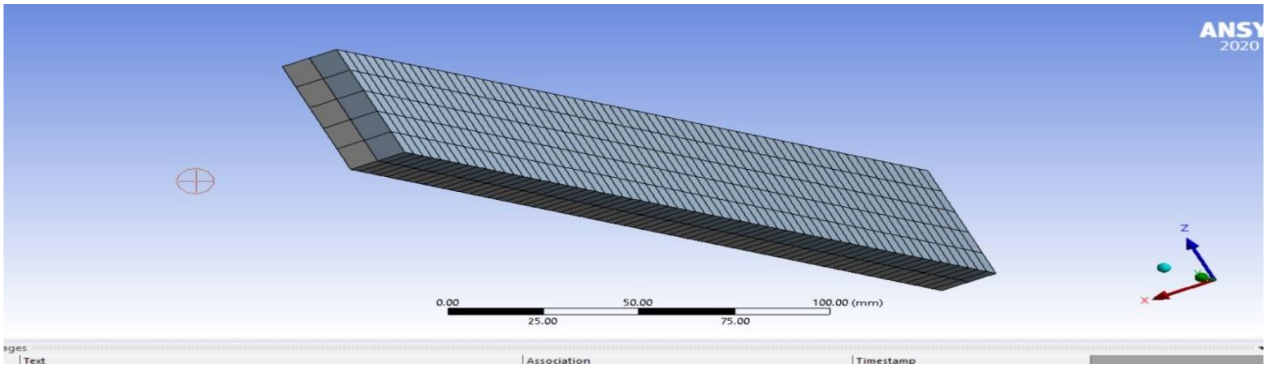
Material	Specimen	Dimension
1. **Steel Alloy**	SM-1	(0.6m X 0.030m X 0.008m)
I. Modulus of Elasticity, E = 210 GPa	SM-2	(0.6m X 0.030m X 0.004m)
II. Density, ρ = 8030 Kg/m3	SM-3	(0.42m X 0.030m X 0.008m)
III. Poisson‘s Ratio, ν = 0.30	SM-4	(0.42m X 0.030m X 0.004m)
2. **Carbon Fiber Reinforced Plastic**	SM-5	(0.6m X 0.030m X 0.008m)
I. Modulus of Elasticity, E = 220 GPa	SM-6	(0.6m X 0.030m X 0.004m)

II. Density, ρ = 1720 Kg/m3	SM-7	(0.42m X 0.030m X 0.008m)
III. Poisson‘s Ratio, ν = 0.33	SM-8	(0.42m X 0.030m X 0.004m)

Material	**Specimen**	**Dimension**
1. **Steel Alloy**	SM-9	(0.6m X 0.030m X 0.016m)
I. Modulus of Elasticity, E = 210 GPa	SM-10	(0.6m X 0.030m X 0.016m)
II. Density, ρ = 8030 Kg/m3	SM-11	(0.42m X 0.030m X 0.024m)
III. Poisson‘s Ratio, ν = 0.30	SM-12	(0.42m X 0.030m X 0.024m)
2. **Carbon Fiber Reinforced Plastic**	SM-13	(0.6m X 0.030m X 0.032m)
I. Modulus of Elasticity, E = 220 GPa	SM-14	(0.6m X 0.030m X 0.032m)
II. Density, ρ = 1720 Kg/m3	SM-15	(0.6m X 0.030m X 0.032m)
III. Poisson‘s Ratio, ν = 0.33	SM-16	(0.6m X 0.030m X 0.032m)

Meshing

By using the spatial element, ANSYS automatically generates the mesh on the beam. 10 nodes that each have three degrees of freedom define the element. It is appropriate for modelling the finite element irregular mesh since it exhibits quadratic shifting behaviour.



3. Post processing:

This menu is helpful to find the output of the problems. Such as –

1. Result summery
2. Failure criteria
3. Plot results
4. List results
5. Result summery
6. Nodal calculation.

We found the output natural frequencies and mode shapes after the application of thismenu which is shown in chapter 5, result and discussion.

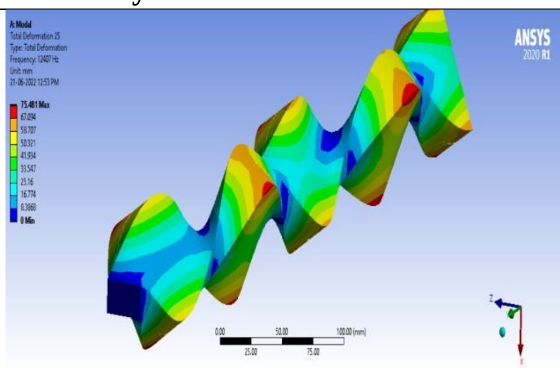
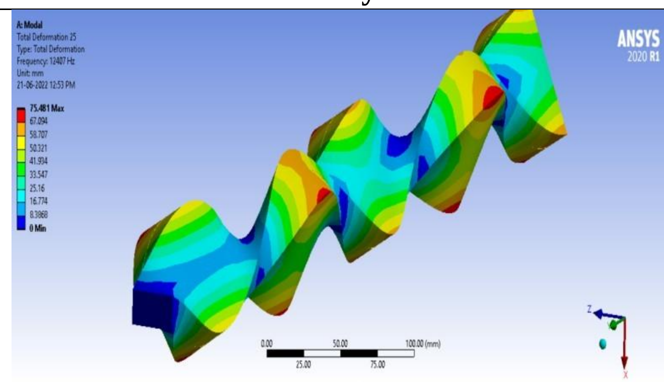
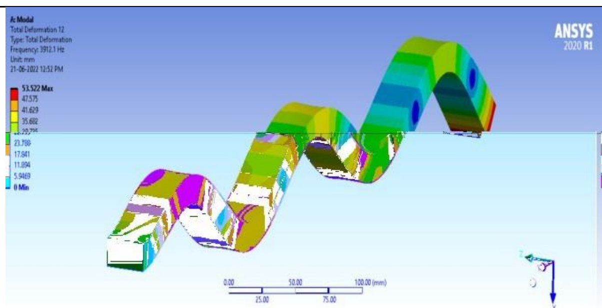
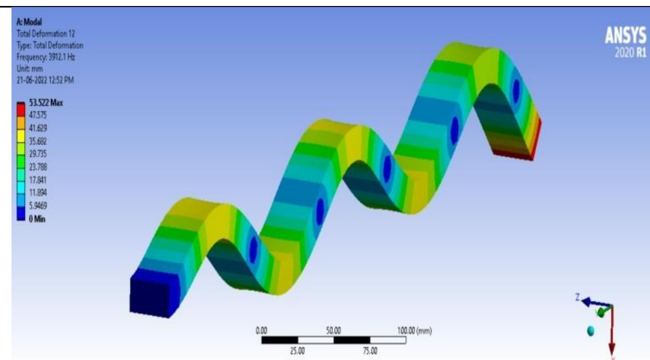
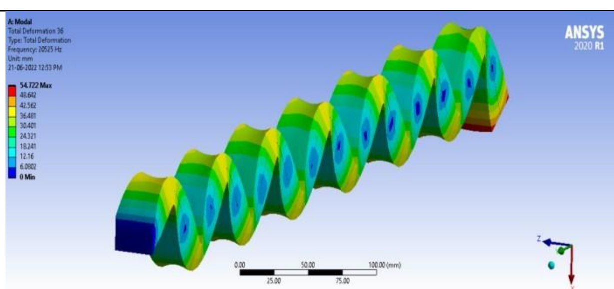
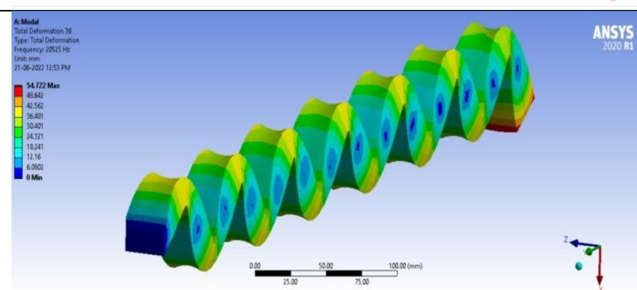
Chapter 5

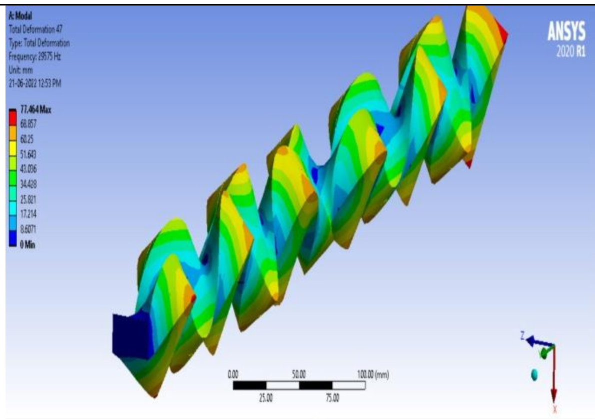
RESULT AND DISCUSSION

RESULT AND DISCUSSION

MODELLING ANALYSIS

ANSYS, Inc. is an engineering modelling and simulation software that offers engineering simulation solution sets in engineering simulation that a design process requires.[1] Here, we are using ANSYS WORKBENCH 14.0 in which modelling of beam is done in geometry component system, material is selected from engineering data library and simulation & analysis is performed in modal analysis system from where we obtained natural frequency and mode shapes for all specimens of both materials

Steel Alloy & Carbon Fiber	**Carbon Fiber & Steel alloy**
	
	
	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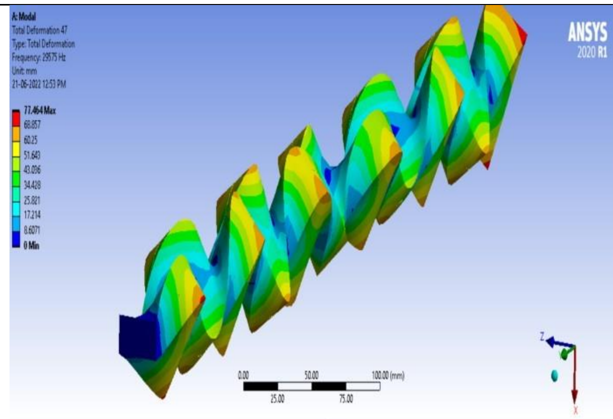
60.25
51.643
43.036
34.428
25.821
17.214
8.6071
0 Min
ANSYS
2020 R1
0.00
50.00
100.00 (mm)
25.00
75.00
Z
X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34.428
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25.00
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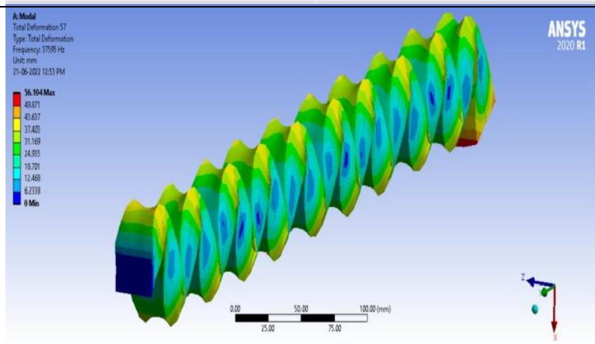
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37.403
31.169
24.935
18.701
12.468
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50.00
100.00 (mm)
25.00
75.00
Z
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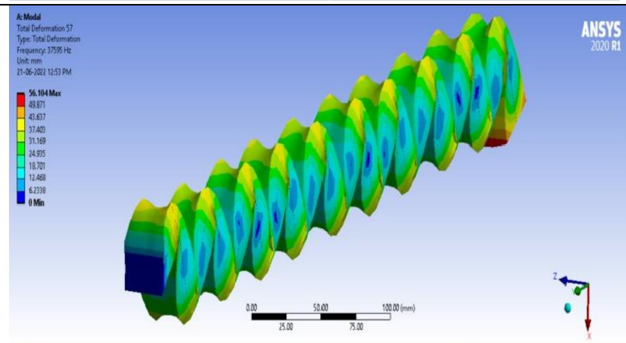
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100.00 (mm)
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Z
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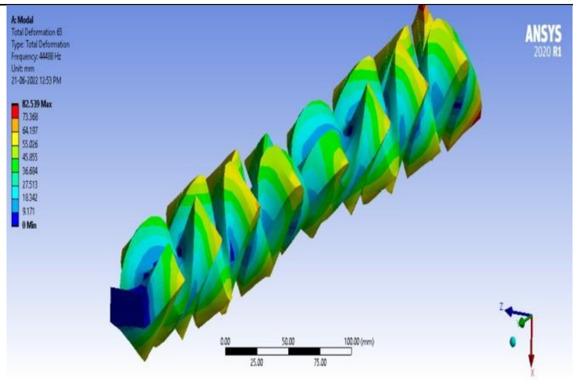
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64.197
55.026
45.855
36.684
27.513
18.342
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0 Min
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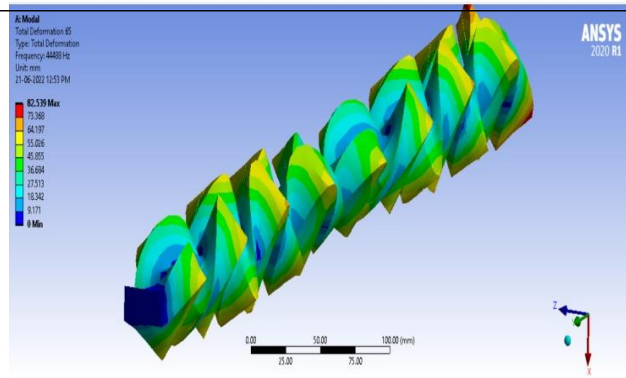
50.00
100.00 (mm)
25.00
75.00
Z
X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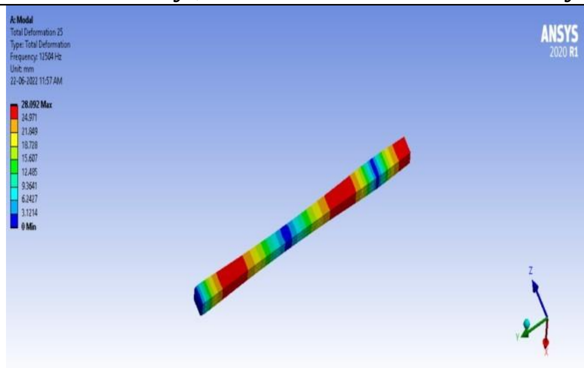
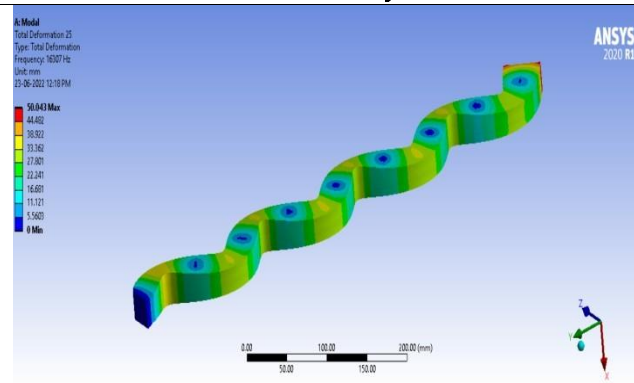
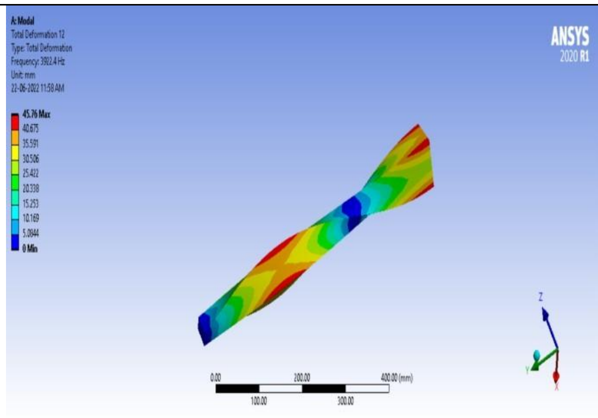
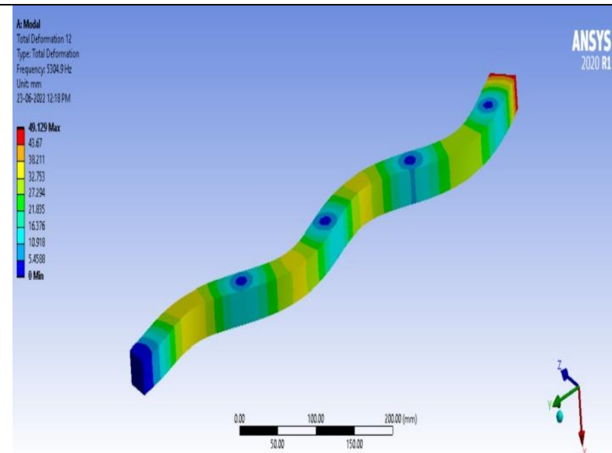
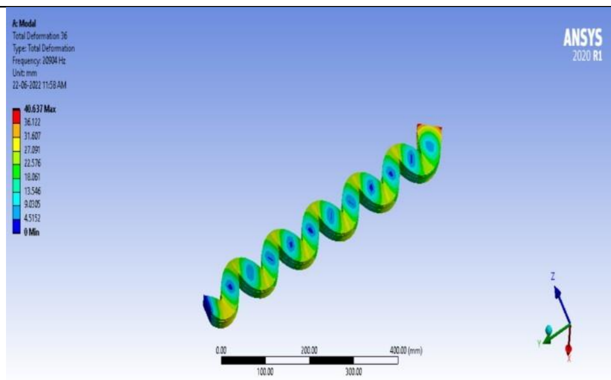
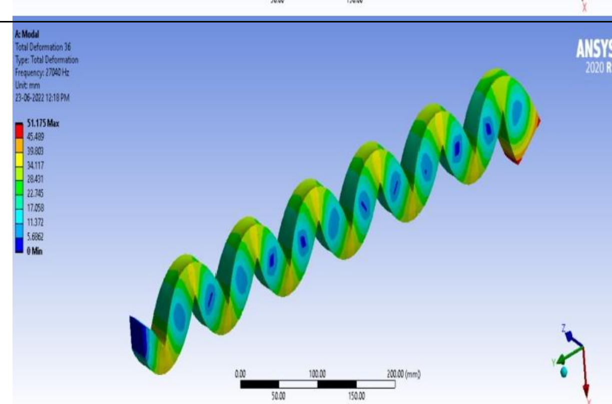
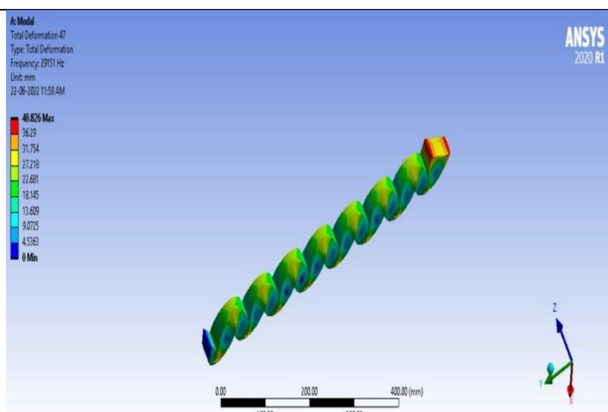
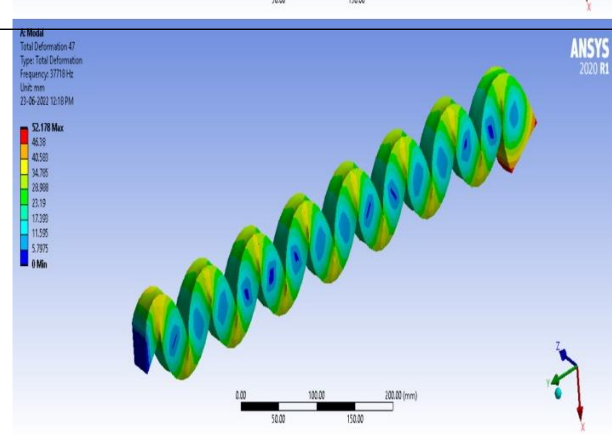
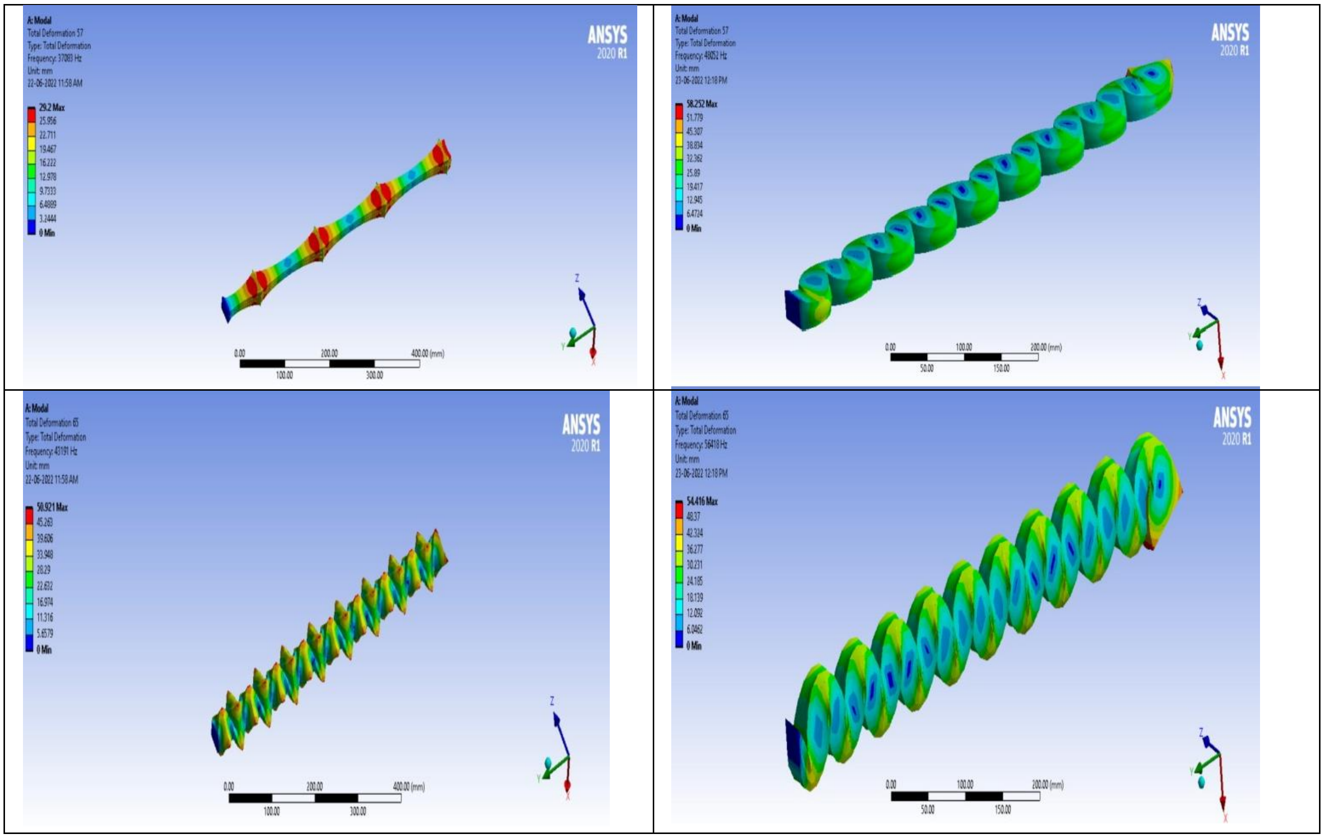
55.026
45.855
36.684
27.513
18.342
9.171
0 Min
ANSYS
2020 R1
0.00
50.00
100.00 (mm)
25.00
75.00
Z
X

Steel Alloy ,Carbon Fiber & Steel Alloy	Carbon Fiber Steel Alloy Carbon Fiber
	
	
	
	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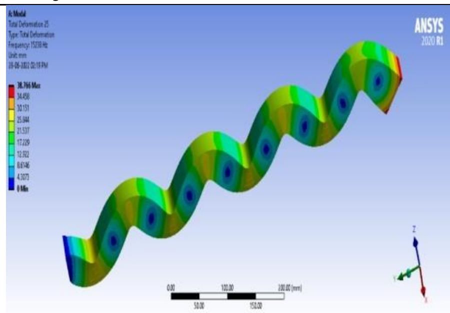
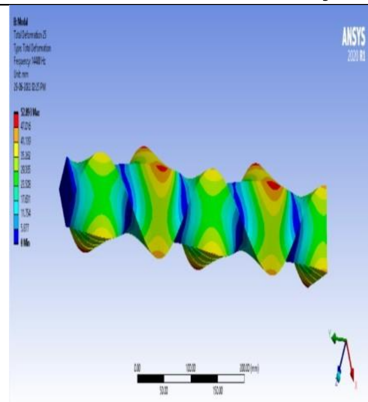
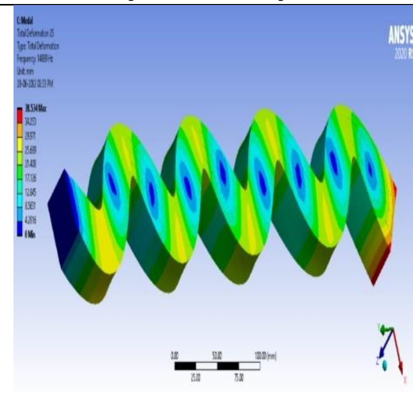
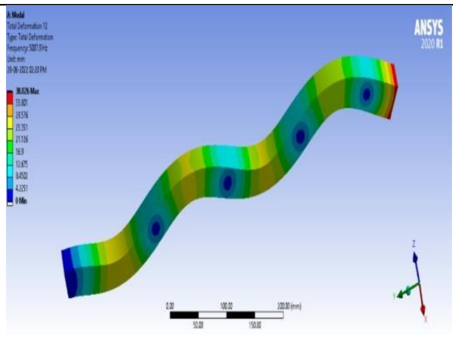
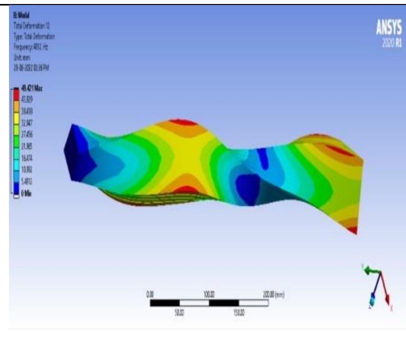
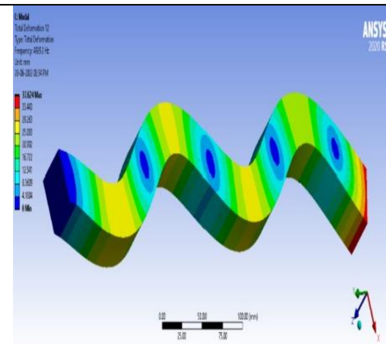
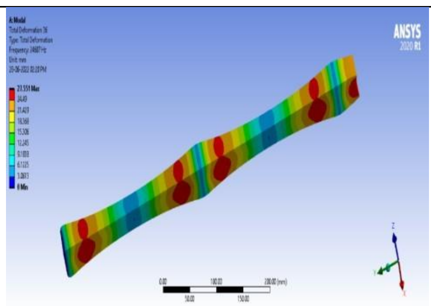
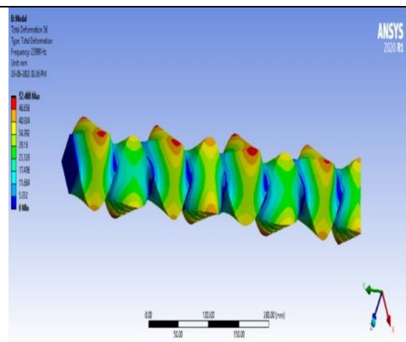
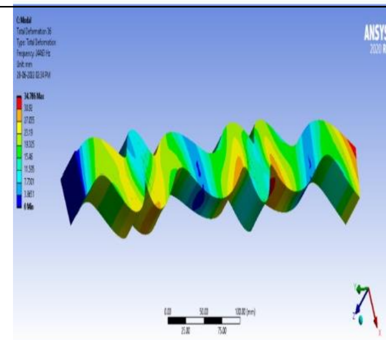
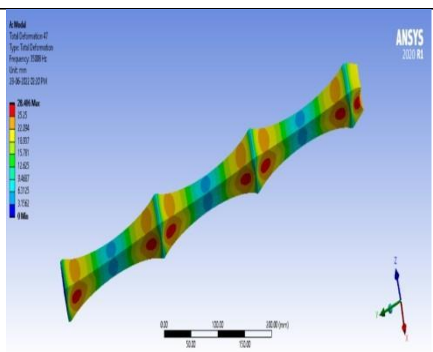
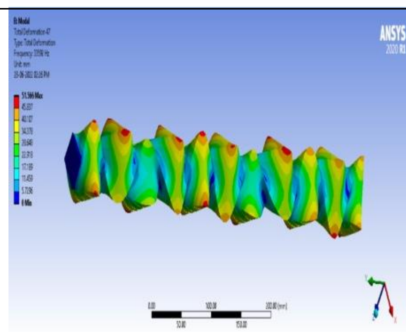
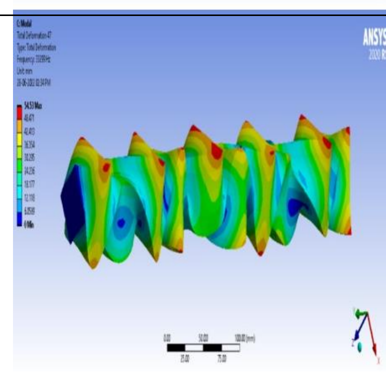
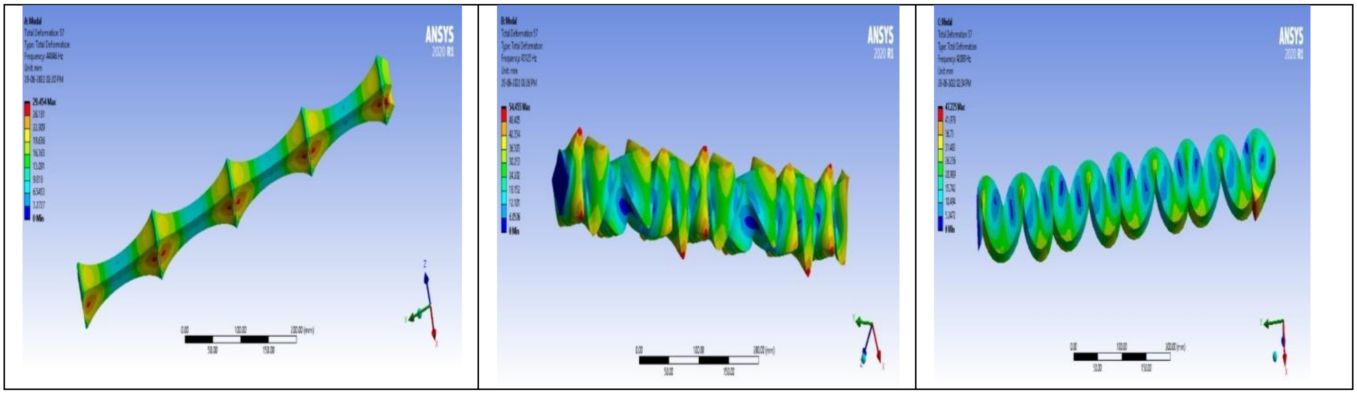
19.467
16.222
12.978
9.7333
6.4889
3.2444
0 Min
ANSYS
2020 R1
0.00
200.00
400.00 (mm)
100.00
300.00
A: Modal
Total Deformation 57
Type: Total Deformation
Frequency: 48052 Hz
Unit: mm
23-06-2022 12:18 PM
58.252 Max
51.779
45.307
38.834
32.362
25.89
19.417
12.945
6.4724
0 Min
ANSYS
2020 R1
0.00
100.00
200.00 (mm)
50.00
150.00
A: Modal
Total Deformation 65
Type: Total Deformation
Frequency: 43191 Hz
Unit: mm
22-06-2022 11:58 AM
50.921 Max
45.263
39.606
33.948
28.29
22.632
16.974
11.316
5.6579
0 Min
ANSYS
2020 R1
0.00
200.00
400.00 (mm)
100.00
300.00
A: Modal
Total Deformation 65
Type: Total Deformation
Frequency: 56418 Hz
Unit: mm
23-06-2022 12:18 PM
54.416 Max
48.37
42.324
36.277
30.231
24.185
18.139
12.092
6.0462
0 Min
ANSYS
2020 R1
0.00
100.00
200.00 (mm)
50.00
150.00

Carbob fiber, steel alloy , steel alloy, Carbon fiber	Carbon Fiber, Steel Alloy, carbon fiber , Steel alloy	Carbon fiber , carbon fiber, steel alloy, steel alloy
		
		
		
		

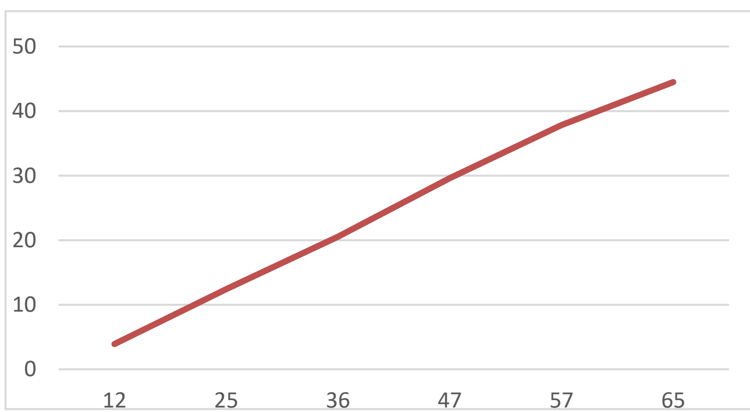


BEAM DEFINED :

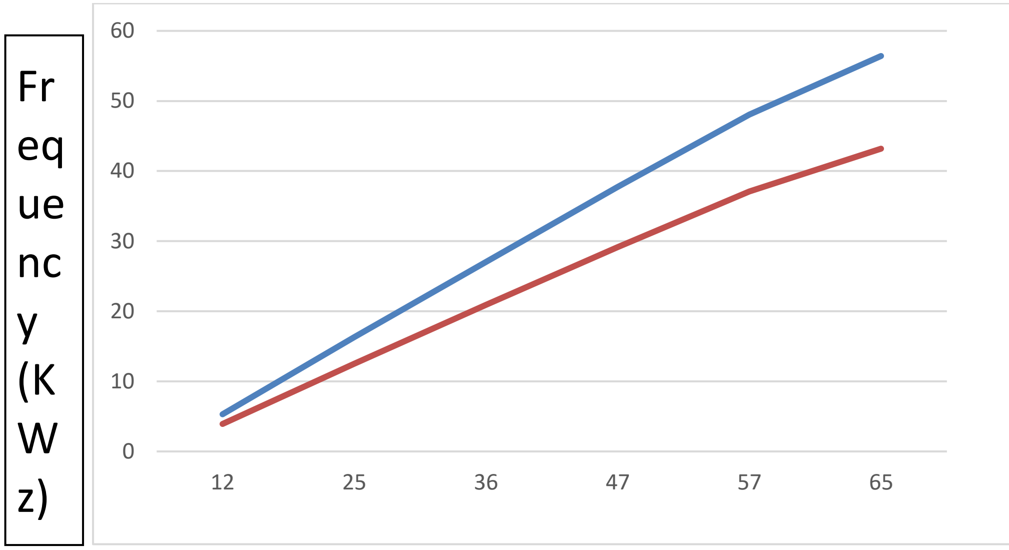
A= STEEL ALLOY + CARBON FIBER REINFORCED PLASTIC
A'= CARBON FIBER REINFORCED PLASTIC + STEEL ALLOY
B= STEEL ALLOY + CARBON FIBER REINFORCED PLASTIC + STEEL ALLOY
B' = CARBON FIBER REINFORCED PLASTIC + STEEL ALLOY + CARBON FIBER REINFORCED PLASTIC
C = CARBON FIBER REINFORCED PLASTIC + STEEL ALLOY + STEEL ALLOY + CARBON FIBER REINFORCED PLASTIC
C' = CARBON FIBER REINFORCED PLASTIC + STEEL ALLOY + STEEL ALLOY + CARBON FIBER REINFORCED PLASTIC
C" = CARBON FIBER REINFORCED PLASTIC + CARBON FIBER REINFORCED PLASTIC + STEEL ALLOY + STEEL ALLOY

Frequency (KWz)

Graph between A&A'



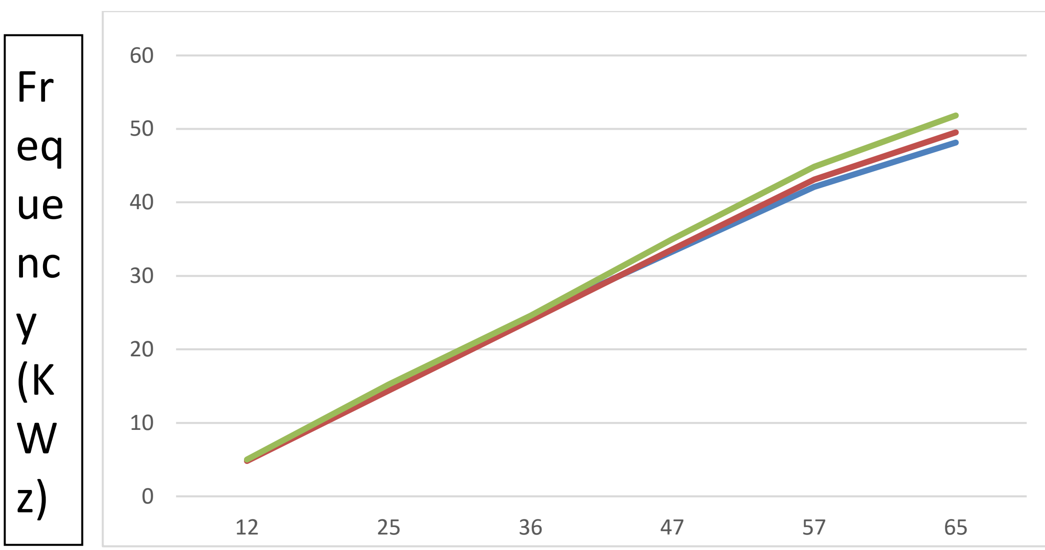
Mode Shape



Mode Shape

Graph between B&B‘

Graph between C,C‘&C‖



Mode Shape

S.No	Mode Shape	S+C (A)	C+S (A’)	S+C+S (B)	C+S+C (B’)	C+S+S+C (C)	C+S+C+S (C’)	C+C+S+S (C”)
1.	12	3.912	3.912	3.922	5.3049	5.0079	4.812	4.9292
2	25	12.407	12.407	12.504	16.307	15.238	14.400	14.889
3	36	20.525	20.525	20.904	27.040	24.607	23.999	24.463
4	47	29.575	29.575	29.151	37.718	35.008	33.592	33.299
5	57	37.795	37.795	37.083	48.052	44.846	43.125	42.089
6	65	44.488	44.488	43.191	56.418	51.829	49.533	48.147

Chapter 6

CONCLUSION AND SCOPE FOR THE FUTUREWORK

CONCLUSION

(1) The model of the cantilever beam was modeled in the programs ANSYS .

(2) Equal mesh was used 10 mm for all the beams ,

(3) it was automatically generated regular mesh

(4) The modal analysis of the Laminated cantilever beam was executed for six mode shapes and their natural frequencies were computed.

(5) Mode shapes of the steel alloy and carbon fiber reinforced plasitic cantilever beam A & A' are identical for both programs.

(6) Mode shapes of the steel alloy ,carbon fiber reinforced plasitic & Steel alloy cantilever beam B & B' are not identical. B' is higher frequency than B beam so B beam is more sutiable

(7) Mode shapes of the steel alloy ,carbon fiber reinforced plasitic & Steel alloy cantilever beam C,C',C" are not identical. C has higher frequency than C' & C" beam,

(8) In case of C' & C" for first few mode C' has less frequencies but as Mode is increases Frequency is also increases compared to C" so C' is sutiable for low mode and C" is suitable for higher mode.

.

Scope for the future work:-

1. An analytical formulation can be derived for modelling the behaviour of laminated composite beams with integrated piezoelectric sensor and actuator. Analytical solution for active vibration control and suppression of smart laminated composite beams can be found. The governing equation should be based on the first-order shear deformation theory (Mindlin plate theory),
2. The dynamic response of an unsymmetrical orthotropic laminated composite beam, subjected to moving loads, can be derived. The study should be including the effects of transverse shear deformation, rotary and higher-order inertia. And also we can provide more number of degree of freedom about 10 to 20 and then should be analyzed by higher order shear deformation theory.
3. The free vibration characteristics of laminated composite cylindrical and spherical shells can be analyzed by the first-order shear deformation theory and a meshless global collocation method based on thin plate spline radial basis function.
4. An algorithm based on the finite element method (FEM) can be developed to study the dynamic response of composite laminated beams subjected to the moving oscillator. The first order shear deformation theory (FSDT) should be assumed for the beam model.
5. The damping behavior of laminated sandwich composite beam inserted with a visco - elastic layer can be derived.

Chapter 7

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REFERENCES

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